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MECHANICS OF STRUCTURES

MECHANICS OF STRUCTURES

[A TEXT-BOOK FOR UNIVERSITY STUDENTS]

By

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WITH 220 DIAGRAMS AND NUMEROUS EXERCISES

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To

My Students

PREFACE

The approach of an average student to the study of this fascinating subject is, somehow, one of unreasoning dread. An attempt has been made in this book to present the principles as simply as possible and to illustrate them with a variety of worked-out examples. A student who goes through the text and takes the trouble of solving the exercises at the end of each chapter will, it is hoped, be better equipped to study with profit the many excellent standard works on the subject.

I am particularly grateful to Mr. R. S. Dighe, B.E.,A.M.I.E., of the Faculty of Technology, for undertaking the arduous and monotonous task of checking up all numerical work, which he has done cheerfully. My thanks are due to Mr. T. N. Joshi, B.Sc. (Eng.), of the Faculty of Technology for making all the sketches required for the blocks which were made by Mr. N. B. Joglekar of the Faculty of Fine Arts, to whom I am grateful.

I am specially indebted to Mr. R. C. Patel of the Charotar Book Stall, Anand, and to Mr. V. B. Priyani, B.E.,A.M.I.E., of the Birla Vishvakarma Mahavidyalaya, who spared no efforts in the tedious task of correcting the proofs, and to Mr. K. A. Damle, B.Sc., Librarian of the Faculty of Technology, who checked the final proofs. My thanks are due to Mr. B. G. Marathe, also of the Faculty of Technology, who took considerable pains in preparing the typescript of the text.

I am indebted to the Syndics of the Universities of Cambridge, London and Bombay, for their kind permission to make use of a few problems set at their University Examinations.

I should like to express my sincere appreciation of the great care and trouble which have been taken by the Anand Press, Anand, in the printing and get-up of this book.

Inspite of all the pains taken by Mr. R. S. Dighe, it is possible that some errors have escaped our attention. I shall be grateful if they are brought to my notice.

Faculty of Technology
including Engineering,
Baroda.
May, 1952

S. B. JUNNARKAR

SECOND IMPRESSION

All the diagrams have been replaced by new ones made by the Prabhat Process Studio, Ahmedabad, to drawings prepared by Mr. N. D. Bhatt, D. M. E. of the Birla Vishvakarma Mahavidyalaya.

Mistakes and misprints, in so far as they were noticed, have been corrected.

Birla Vishvakarma Mahavidyalaya,
Vallabh Vidyanagar, ANAND.
July, 1953

S. B. JUNNARKAR

THIRD IMPRESSION

The author is gratified that the third impression of this book was necessary within a very short space of time. The steady demand for the book from students all over the country seems to justify the object with which it was written. The author has had no time to undertake any revision but has to be content with arranging the matter in a more presentable form.

Birla Vishvakarma Mahavidyalaya,
Vallabh Vidyanagar, ANAND.
February, 1955

S. B. JUNNARKAR

SECOND EDITION

In this edition, about eighty problems have been added for practice to the exercises given at the ends of chapters. The questions selected from Bombay, Cambridge, and London University examinations have now been marked (B. U.), (C. U) and (L. U.) respectively.

I am grateful to Mr. M. A. Chitale, B.E. for checking the answers to the additional problems.

POONA.
February, 1957

S. B. JUNNARKAR

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MECHANICS OF STRUCTURES

CHAPTER I

SIMPLE STRESSES AND STRAINS

1. Stress: Every material is elastic so that when an external system of forces acts on a body, it is deformed. As it undergoes deformation, it sets up resistance to deformation. This resistance increases as the deformation increases and when full resistance to the external forces is put up, the process of deformation stops. If the body is unable to put up the full resistance, the process of deformation continues until failure takes place. The deformation is called "*Strain*" and the accompanying resistance to deformation is called "*Stress*".

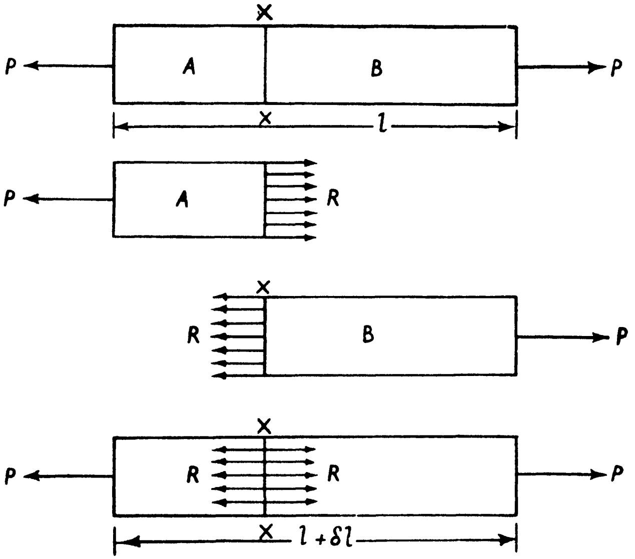


FIG. 1

2. Simple Stresses and Strains: Consider a bar of uniform section *a* and length *l*, which is subjected to two axial pulls *P*.

Imagine the bar to be divided into two portions A and B by a section XX. On A, there is the external force P and unless B comes to its assistance at the section XX, it must move. The resistance to the external force P acting on A is supplied by B while A stretches. When this reaches the full value R required to overcome P, the deformation of A ceases. Likewise, the external force P acting on B is resisted by A coming to its assistance and supplying it with the necessary resistance R, when the deformation of B also stops.

When the section XX — and, for that matter, every section — has put up full resistance to the external pulls P, the length l of the bar will have finally altered to $l + \delta l$. The resistance R is the stress and the deformation δl is the strain. The intensity of stress per unit area of resisting section is obviously $p = \frac{R}{a} = \frac{P}{a}$, since $R = P$ if the axial pull P is gradually applied. Strain is usually measured by the deformation per unit length. The strain e therefore is equal to $\frac{\delta l}{l}$.

There are three simple states of stress set up by a resisting section.

(1) *Tensile stress:* When the section is subjected to axial pulls P acting normally across the section. The corresponding strain is called tensile strain. If l is the length of the bar and A its sectional area, the tensile stress $p - \frac{P}{A}$ and tensile strain $e = \frac{\delta l}{l}$, where δl is the elongation.

(2) *Compressive stress:* When the section is subjected to axial pushes P acting normally across the section. The corresponding strain is called compressive strain. Compressive stress $p = \frac{P}{A}$. In this case the original length l will shorten by δl and the compressive strain $e = \frac{\delta l}{l}$.

(3) *Shear stress:* When the section is subjected to two equal and opposite forces P acting tangentially across the

resisting section as in the case of the section of a rivet holding together two plates in tension. The intensity of shear stress $q = \frac{R}{A}$ where R is the resistance to P set up by section XX and A is its area of cross-section. If P is gradually applied $R = P$ and $q = \frac{P}{A}$.

hea. ress

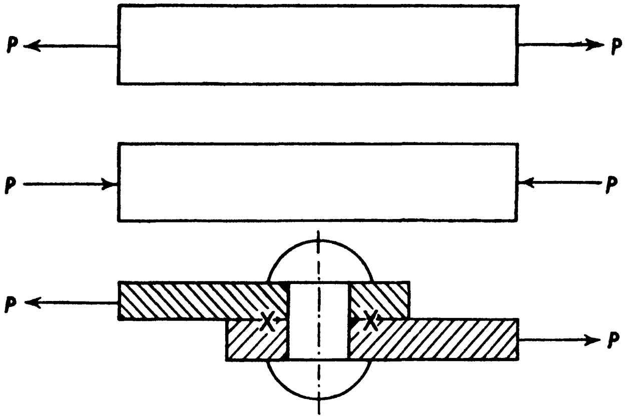


FIG. 2

The corresponding deformation or shear strain ϕ is measured by the angular deformation accompanying the shear stress.

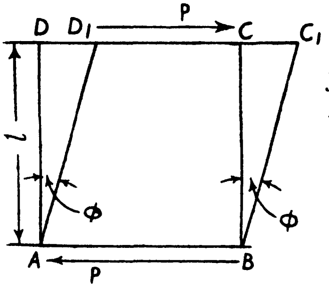


FIG. 3

Consider a cube of side l subjected to equal and opposite forces P across the top and bottom faces. If the bottom face AB be fixed, the face $ABCD$ will be distorted to ABC_1D_1, through angle ϕ.

By definition, strain is deformation per unit length. Shear strain is, therefore $\frac{CC_1 \text{ or } DD_1}{l} = \frac{CC_1 \text{ or } DD_1}{BC \text{ or } AD} = \phi$ if CC_1 may be looked upon as an elementary arc of a circle with centre B and radius BC, subtending the angle ϕ at the centre B. ϕ, the angular deformation accompanying a shear stress q is thus the measure of the shear strain.

It will be noted that the intensity of stress being resistance per unit area, its units will be lb. or ton per sq. inch or, lb. or ton per sq. foot, depending on the units of force and area used.

The corresponding strain, whether tensile, compressive or shear is always a number. The shear strain ϕ, being in radians, is a number.

3. Elastic Limit: A material is said to be perfectly elastic if the deformation due to external loading entirely disappears on removal of the load. That the resistance or stress should disappear on removal of the external action is obvious; but the deformation undergone need not entirely disappear. For every material, a limiting value of load for a given resisting section is found, upto and within which, the resulting strain entirely disappears when the load is removed. The value of the intensity of stress corresponding to this limiting load is known as the Elastic Limit. If loading beyond the elastic limit takes place, the structure of the materials is damaged and so there is a residual strain on removal of the stress. Such residual strain is called a permanent set. Beyond the elastic limit the material gets into the plastic stage but within it, it behaves like a perfect spring, regaining its original dimensions on removal of the loading.

4. Hooke's Law: It is found that when a material is loaded within its elastic limit, the stress or resistance set up is proportional to the strain or deformation produced. Expressed as an equation, $\dfrac{\text{stress}}{\text{strain}} = \text{a constant}$.

Referring to the fundamental stresses viz., tension, compression or shear, within the elastic limit,

(1) $\dfrac{p}{e} = E$, a constant which is called the modulus of Elasticity for tensile and compressive stresses. It is also

known as Young's modulus after Young who first determined its magnitude.

(2) $\frac{q}{\phi} = N$, a constant which is called the modulus of Rigidity or Shear modulus, for shear stresses. It is usually also denoted by letters *C* and *G*.

The values of *E* and *N* are naturally not the same for a given material, because the nature of stress and strain is different. For a given material, there is, however, a definite relation between the two constants which will be studied later. The units for these constants are the same as stress units.

Example 1.

A steel rod, 1″ diameter and 10 ft. long is subjected to an axial pull of $5\frac{1}{2}$ tons. If E = 13000 tons/in^2., calculate the elongation of the bar.

Area of resisting section $A = \frac{\pi}{4} = \cdot 7854$ in^2.

Intensity of stress $p = \frac{P}{A} = \frac{5\cdot 5}{\cdot 7854} = 7$ tons/in^2.

Strain $e = \frac{p}{E} = \frac{7}{13000}$

$\therefore \quad \delta l = e.l = \frac{7}{13000} \quad 120 = \cdot 0646$ inch.

Example 2.

The steam pressure in a locomotive boiler is 200 lb/in^2. The stay bolts which connect the sides of the firebox with the end plate of the boiler are placed 4″ centre to centre vertically and horizontally. Calculate the pull on a stay bolt and the necessary diameter if the permissible stress in the material is 14000 lb/in^2.

Each bolt is in charge of 4″ × 4″ plate area.

$\therefore$ The pull on each bolt $P = 200 \times 4 \times 4 = 3200$ lb-wt.

The necessary area of bolt-section,

$$A = \frac{P}{p} = \frac{3200}{14000} = \frac{8}{35} = \frac{\pi}{4}d^2$$

$\therefore \quad d^2 = \cdot 291.$

The minimum diameter d of each stay bolt $= \cdot 54''$, say, $\frac{5}{8}''$.

5. Loading beyond the Elastic Limit: If a piece of ductile material is subjected to a gradually increasing pull, it is found that the resulting strain increases proportionally to the corresponding stress, as expected, upto the elastic limit. When the elastic limit is passed, the straight line relation between stress and strain ceases, the strain increasing more quickly as the stress increases. In some metals like wrought iron and steel, there is a marked breakdown at a stress a little above the elastic limit. This is known as the Yield Point. Figure 4 shows a typical stress-strain diagram for a ductile material in a tensile test carried to destruction. At the maximum stress represented by the point C in the diagram, local yielding takes place and the specimen begins to form a "waist". When this happens, the load necessary to break the specimen is less than the maximum load. The stress is therefore reduced until fracture takes place at the stress represented by the point D. This is the breaking stress.

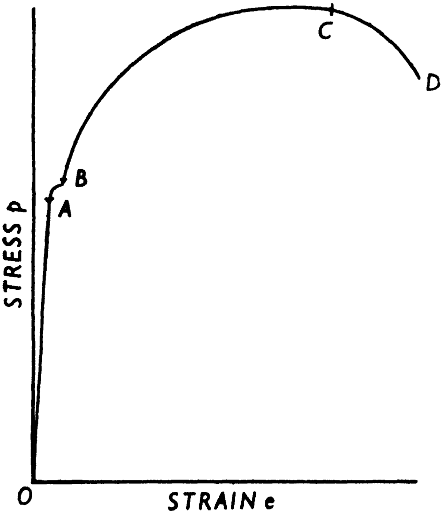


FIG. 4

6. Ultimate stress. Working stress: The maximum or ultimate stress is measured by the maximum load reached divided by the area of the original cross-section of the test piece.

The greatest resistance or stress which a material is called upon to put up—in other words, the stress which a material will be worked to—is called its working stress. It is naturally much less than the ultimate stress and is actually well within the elastic limit.

The ratio of the ultimate stress and the working stress is called the Factor of Safety. These definitions apply to compressive and shear stresses also.

Example 3.

A hollow cast iron column 8″ external diameter, thickness of metal 1″ is subjected to an axial load. If the ultimate crushing stress for the material is 36 tons/in^2., calculate the safe load for the column, using a factor of safety of 6.

Area of section $A = \frac{\pi}{4}(8^2 - 6^2) = \frac{\pi}{4} \times 14 \times 2 = 22 \text{ in}^2.$

Crushing load for the column $= 36 \quad 22 = 792$ tons.
Safe load $= \frac{792}{6} = 132$ tons.

Example 4.

If the ultimate shear stress for mild steel is 20 tons/in^2., find the force required to punch a $\frac{1}{2}$″ hole in a mild steel plate $\frac{3}{8}$″ thick. Calculate the compressive stress on the punch.

To punch a hole of diameter d, in a plate of thickness t, the necessary force on the punch $P = \pi d.t.f_s$ where $\pi d.t$ is the plate-area resisting shear and f_s is the ultimate shear stress for the plate-material.

$P = \pi \times \frac{1}{2} \times \frac{3}{8} \times 20 = 11{\cdot}78$ tons.

The compressive stress on the punch $= \dfrac{P}{A}$, where A is the area of cross-section of the punch.

$$\therefore \quad \frac{P}{A} = \frac{11{\cdot}78}{{\cdot}1963} = 60 \text{ tons/in}^2.$$

7. Lateral strain. Poisson's Ratio: *Any direct stress is accompanied by a strain in its own direction and an opposite kind of strain in every direction at right angles to it.* All axial strain, tensile or compressive, is accompanied by lateral strain. Thus the length of a tie-bar will increase under an axial pull but its cross-section will decrease. A column under an axial load will shorten but its section will slightly increase. This lateral strain or strain of the sides, is a fraction of the linear strain and, within the elastic limit, bears a constant relation to the linear strain.

Thus, $\dfrac{\text{lateral strain}}{\text{linear strain}} = \text{constant}$, within the elastic limit.

This constant is known as Poisson's Ratio and is usually denoted by $\frac{1}{m}$. The value of m lies between 3 and 4 for most metals. In other words, Poisson's Ratio for most metals is of the order of ·25 to ·34.

We may now work out the changes in all the dimensions of a bar subjected to a simple pull or push. If the linear strain of the bar is e, the lateral strain will be $\frac{1}{m}$ e. If the cross-section of the bar is rectangular—b wide t thick—the changes in the width and thickness will be given by:

$$\frac{\delta b}{b} = \frac{\delta t}{t} = \text{lateral strain} = \frac{1}{m} \times e.$$

If the bar is round in section, of diameter d, the change in diameter is given by:

$$\frac{\delta d}{d} = \text{lateral strain} = \frac{1}{m} \times e.$$

Since all the dimensions are slightly altered under the axial loading, the volume of the bar will also alter.

For a rectangular bar, the volume,

$$V = l.b.t.$$

$$\therefore \quad \delta V = b.t.\delta l + t.l.\delta b + l.b.\delta t.$$

And $\frac{\delta V}{V} = \frac{\delta l}{l} + \frac{\delta b}{b} + \frac{\delta t}{t}$.

The ratio of the change of the volume and the original volume is thus equal to the algebraic sum of the strains of the three sides.

Thus $\frac{\delta V}{V} = e - \frac{1}{m} \times e - \frac{1}{m} \times e = e(1 - \frac{2}{m})$

remembering that the lateral strain is of the opposite sign to the linear strain.

If the bar is round, of diameter *d*, the volume

$$V = \frac{\pi}{4}d^2.l$$

$$\delta V = \frac{\pi}{4}d^2.\delta l + 2\ \frac{\pi}{4}d.l.\delta d$$

and $\frac{\delta V}{V} = \frac{\delta l}{l} + \frac{2\delta d}{d} = e - \frac{2}{m} \times e = e\,(1 - \frac{2}{m})$, as before.

The slight alteration in volume that takes place can thus be easily calculated.

When a bar is subjected to a gradually applied axial pull or push, work is being done on it and, within the elastic limit, is stored by it. The work done or stored is given by the product of the average force and the change of length i.e. by $\frac{1}{2}P \times \delta l$.

Example 5.

A steel flat 8″ wide, $\frac{3}{8}$″ thick and 20 feet long carries a pull of 22 tons. Find the extension in length and the contraction in width and thickness under the pull. Take Poisson's Ratio as 0·3 and E = 13500 tons/in². Calculate also the change in volume.

$$p = \frac{P}{A} = \frac{22}{8 \times \frac{3}{8}} = 7{\cdot}33 \text{ tons/in}^2.$$

$$e = \frac{p}{E} = \frac{7{\cdot}33}{13500} = {\cdot}000543.$$

The stretch of the bar $\delta l = e.l = {\cdot}000543 \times 240 = {\cdot}13''$.

Lateral strain $= \frac{1}{m}.e = {\cdot}000163$.

$$\therefore \quad \delta b = 8 \times {\cdot}000163 = {\cdot}001304''$$

$$\delta t = \tfrac{3}{8} \times {\cdot}000163 = {\cdot}000061''$$

$$\frac{\delta V}{V} = e\left(1 - \frac{2}{m}\right) = {\cdot}000543\ (1 - {\cdot}3 \times 2) = {\cdot}000543 \times {\cdot}4$$

$$= {\cdot}000217$$

$$\therefore \quad \delta V = {\cdot}000217 \times 8 \times \tfrac{3}{8} \times 240 = {\cdot}156 \text{ cub. inch.}$$

Example 6.

A steel bar $\frac{7}{8}''$ in diameter 10 feet long is subjected to an axial pull of 6 tons. If $E = 13000$ tons/in^2., and $m = 4$, calculate the alteration in (i) the length (ii) diameter (iii) volume. Calculate also the work done in stretching the bar.

$$p = \frac{P}{A} = \frac{6}{0{\cdot}6} = 10 \text{ tons/in}^2.$$

$$e = \frac{p}{E} = \frac{10}{13000} = \frac{1}{1300}.$$

The stretch of the bar $\delta l = e.l = \frac{1}{1300} \times 120 = {\cdot}0923''$.

Lateral strain $= \frac{1}{m} \times e = \frac{1}{4} \times \frac{1}{1300} = \frac{1}{5200}$.

$\therefore$ Contraction of diameter $\delta d = \frac{1}{5200} \times \frac{7}{8} = {\cdot}000168''$.

$$V = \frac{\pi}{4} d^2.l = {\cdot}6 \times 120 = 72 \text{ cub. inches.}$$

$$\frac{\delta V}{V} = e\left(1 - \frac{2}{m}\right) = \frac{1}{1300}\left(1 - \tfrac{2}{4}\right) = \frac{1}{2600}$$

$$\delta V = \frac{1}{2600} \times 72 = {\cdot}0277 \text{ cub. inch.}$$

The work done in stretching the bar

$$= \frac{P}{2} \times \delta l = \tfrac{1}{2} \times 6 \times {\cdot}0923 = {\cdot}2769 \text{ in-ton.}$$

8. Bars of varying section: If a bar is made up of different lengths of different sectional areas, the stress in each section and the deformation of each length under an axial load can be worked out as usual. The total change of length will be the sum of the changes of different lengths.

Consider a bar of length l made up of l_1, l_2 and l_3 with sectional areas of A_1, A_2 and A_3, under an axial pull P.

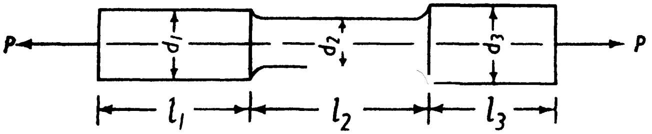


FIG. 5

The stresses p_1, p_2 and p_3 in the respective sections are given by:

$$p_1 = \frac{P}{A_1};\ p_2 = \frac{P}{A_2};\ p_3 = \frac{P}{A_3},\ \text{or } P = p_1A_1 = p_2A_2 = p_3A_3.$$

The strains of different lengths are:

$$e_1 = \frac{p_1}{E};\ e_2 = \frac{p_2}{E};\ e_3 = \frac{p_3}{E}.$$

The changes in the lengths are:

$$\delta l_1 = e_1l_1;\ \delta l_2 = e_2l_2;\ \delta l_3 = e_3l_3.$$

The total change of length $\delta l = \delta l_1 + \delta l_2 + \delta l_3$.

Example 7.

A wrought-iron bar, 24 feet long is $1\frac{1}{8}''$ in diameter for 6 feet of its length, 1" in diameter for 12 feet and $1\frac{1}{4}''$ in diameter for the remaining length. It is subjected to an axial compressive load of 10000 lb-wt. If $E = 28 \times 10^6$ lb/in²., calculate the change in the length of the bar and the amount of energy stored in it.

$$A_1 = \frac{\pi}{4} \times 1{\cdot}125^2 = 0{\cdot}994 \text{ sq.inch};\ l_1 = 72''$$

$$A_2 = \frac{\pi}{4} \times 1 = 0{\cdot}7854 \text{ sq.inch};\ l_2 = 144''$$

$$A_3 = \frac{\pi}{4} \times 1{\cdot}25^2 = 1{\cdot}227 \text{ sq.inch};\ l_3 = 72''$$

$$p_1 = \frac{P}{A_1} = \frac{10000}{0{\cdot}994} = 10060 \text{ lb/in}^2.$$

$$e_1 = \frac{p_1}{E} = \frac{10060}{28 \times 10^6} = \cdot 00036$$

$$p_2 = \frac{P}{A_2} = \frac{10000}{0 \cdot 7854} = 12740 \text{ lb/in}^2.$$

$$e_2 = \frac{p_2}{E} = \frac{12740}{28 \times 10^6} = \cdot 000455$$

$$p_3 = \frac{P}{A_3} = \frac{10000}{1 \cdot 227} = 8150 \text{ lb/in}^2.$$

$$e_3 = \frac{p_3}{E} = \frac{8150}{28 \times 10^6} = \cdot 00029$$

$$\delta l_1 = e_1 l_1 = \cdot 00036 \times 72 = \cdot 026''$$

$$\delta l_2 = e_2 l_2 = \cdot 000455 \times 144 = \cdot 066''$$

$$\delta l_3 = e_3 l_3 = \cdot 00029 \times 72 = \cdot 021''.$$

The change in length $\delta l = \delta l_1 + \delta l_2 + \delta l_3$

$= \cdot 026 + \cdot 066 + \cdot 021$

$= 0 \cdot 113''.$

The energy stored in the bar $= \frac{1}{2} P \times \delta l$

$= \frac{1}{2} \times 10000 \times \cdot 113$

$= 565$ inch-lbs.

$= 47 \cdot 1$ ft-lbs.

9. Bar of uniformly tapering section: Let P be the axial pull on a bar of length l whose diameter varies uniformly from d_1 at one end to d_2 at the other.

Consider an element δx of the bar at a distance x from the larger end. Let d_x be the diameter at distance x from the larger end.

Then $d_x = d_1 - (d_1 - d_2) \frac{x}{l} = d_1 - kx$, where $k = \frac{d_1 - d_2}{l}$.

The area A_x of section is $= \frac{\pi}{4} . d_x^2 = \frac{\pi}{4}(d_1 - kx)^2$.

The intensity of stress set up by the section

$$p_x = \frac{P}{A_x} = \frac{4P}{\pi (d_1 - kx)^2}$$

The strain $e_x = \frac{p_x}{E} = \frac{4P}{\pi E (d_1 - kx)^2}$

The elongation of the element δx due to p_x which is sensibly uniform if δx is sufficiently small, then, is given by,

$$\frac{\delta\cdot\delta x}{\delta x} = e_x = \frac{4P}{\pi E(d_1 - kx)^2}$$

$$\therefore \quad \delta\cdot\delta x = \frac{4P.\delta x}{\pi E(d_1 - kx)^2}.$$

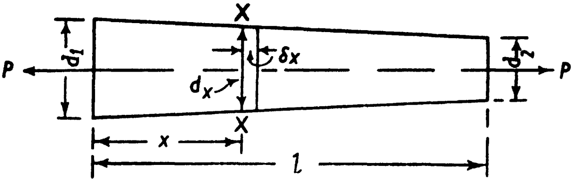


FIG 6

Summing up the stretches of all these elementary lengths δx from one end to the other, we get,

$$\sum_{0}^{l} \delta.\delta x = \int_{0}^{l} \frac{4P.dx}{\pi E(d_1 - kx)^2}$$

The elongation
$$\delta l = \frac{4P}{\pi Ek}\left\{\frac{1}{d_1 - kx}\right\}_{0}^{l}$$
$$= \frac{4P}{\pi Ek}\left\{\frac{1}{d_1 - kl} - \frac{1}{d_1}\right\}$$
$$= \frac{4P}{\pi Ek}\left\{\frac{1}{d_2} - \frac{1}{d_1}\right\} = \frac{4P \times l \times (d_1 - d_2)}{\pi E\,(d_1 - d_2)\; d_1 d_2}$$
$$= \frac{4Pl}{\pi E.d_1 d_2}$$

If the bar had been of uniform diameter d throughout,

$$A = \frac{\pi}{4}d^2;\; p = \frac{P}{A} = \frac{4P}{\pi d^2};\; e = \frac{p}{E} = \frac{4P}{\pi E.d^2}$$

and $\delta l = e.l = \dfrac{4Pl}{\pi E d^2}.$

If the bar uniformly tapers from d_1 to d_2, the d^2 in this expression changes to $d_1 d_2$.

Example 8.

A brass rod, circular in section, tapers from 1″ diameter to ½″ diameter in a length of 10 inches. How much will this length *increase under an axial pull of 1200 lb-wt. if* $E = 14 \times 10^6$ *lb/in²*.

$$\delta l = \frac{4Pl}{\pi E.d_1 d_2} = \frac{4 \times 1200 \times 10}{\frac{22}{7} \times 14 \times 10^6 \times 1 \times \frac{1}{2}} = \frac{24}{1100} = \cdot 0022''.$$

10. Bars of composite section: When the cross-section of the bar is made up of different materials—say two — whose sectional areas are A_1 and A_2 and moduli of Elasticity are E_1 and E_2 respectively, let p_1 and p_2 be the stresses set up in the respective materials under an axial load P on the composite bar. If δl is the deformation in a length l of the bar,

$$e = \frac{\delta l}{l} = \frac{p_1}{E_1} = \frac{p_2}{E_2} \ldots\ldots\ldots\ldots (1)$$

Also $P = P_1 + P_2 = p_1 A_1 + p_2 A_2 \ldots\ldots\ldots\ldots (2)$

With the help of these two equations, we can solve for p_1 and p_2, and the portion of the load resisted by each material.

Example 9.

A mild steel bar 1″ in diameter and 12″ long is enclosed in a brass tube whose external diameter is 1½″ and internal diameter is 1″. The composite bar is then subjected to an axial pull of 5 tons. If E for steel is 30×10^6 *lb/in². and for brass* 14×10^6 *lb/in²., find the stresses in the bar and tube, the extension of the bar and the work done in stretching.*

The area of steel section $A_s = \cdot 7854$ in².

The area of brass section $A_b = \frac{\pi}{4}(1 \cdot 5^2 - 1) = \cdot 9815$ in².

$$\frac{p_s}{E_s} = \frac{p_b}{E_b} \therefore p_s = \frac{E_s}{E_b} \times p_b = \frac{30 \times 10^6}{14 \times 10^6} p_b = 2 \cdot 14 p_b.$$

Also $P_s + P_b = P$

or $p_s A_s + p_b A_b = P$

$\therefore$ $\cdot 7854\, p_s + \cdot 9815\, p_b = 5.$

Substituting $2 \cdot 14\, p_b$ for p_s we get

$p_b\,(1 \cdot 68 + \cdot 9815) = 5$

or $2 \cdot 66\, p_b = 5$

$\therefore\ p_b = 1{\cdot}88$ tons/in².
and $p_s = 4{\cdot}03$ tons/in².

The load resisted by the brass tube $= p_bA_b = 1{\cdot}88 \times {\cdot}9815 = 1{\cdot}845$ tons, the balance of 3·155 tons being resisted by the steel section.

The strain $e = \dfrac{p_b}{E_b}$ or $\dfrac{p_s}{E_s} = \dfrac{4{\cdot}03 \times 2240}{30 \times 10^6} = {\cdot}0003$.

The extension of the bar $= e.l = {\cdot}0003 \times 12 = {\cdot}0036''$.

Work done in stretching $= \frac{1}{2}P.\delta l = \frac{1}{2} \times 5 \times {\cdot}0036$
$= {\cdot}009$ inch-ton.

Example 10.

A reinforced concrete column of square section, 12″ sides, has four reinforcing bars $1\frac{1}{8}''$ diameter, one in each corner, with its centre 3″ from the edges. Find the safe central load on the column if concrete can be stressed to 600 lb/in². What is the corresponding stress in the steel reinforcement and what portion of the load is borne by it? Take the modular ratio of steel and concrete $\dfrac{E_s}{E_c} = 18$.

$A_s = 4 \times {\cdot}994 = 4$ in².

$A_c = 12 \times 12 - A_s = 144 - 4 = 140$ in².

$\dfrac{p_s}{E_s} = \dfrac{p_c}{E_c}\ \therefore\ p_s = \dfrac{E_s}{E_c}\ p_c = 18p_c = 10800$ lb/in²., when $p_c = 600$ lb/in².

Safe central load $= p_cA_c + p_sA_s$
$= 600 \times 140 + 10800 \times 4$
$= 84000 + 43200$
$= 127200$ lb-wt.

Of this, the reinforcing bars carry 43200 lb-wt.

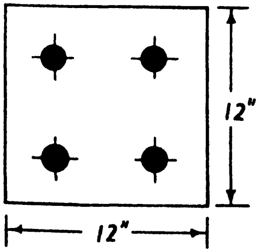


FIG. 7

Example 11.

Two vertical rods — one of steel and the other of bronze — are each rigidly fastened at the upper end at a horizontal distance of 30″ apart. Each rod is 8 feet long and $\frac{1}{2}''$ diameter. A horizontal cross-

piece connects the lower ends of the bars. Where should a load of 2000 lb-wt. be placed on the cross-piece so that it remains horizontal after being loaded? Estimate the stress in each rod. $E_s = 30 \times 10^6$ *lb/in*^{2}. $E_b = 16 \times 10^6$ *lb/in*^{2}.

Let the distance of the load P of 2000 lb-wt. on the cross-piece be x'' from the centre of the steel rod and let P_s and P_b be the loads borne by the respective rods.

Then $P = P_s + P_b$

$= p_sA_s + p_bA_b$, p_s and p_b being the stresses and A_s and A_b being the areas of sections of the rods.

In this case $A_s = A_b = \cdot 1963$ in^2.

$\therefore \quad \cdot 1963\ (p_s + p_b) = 2000.$

Also, since the cross-piece is to remain horizontal after loading, the strain of each rod must be the same.

$$\therefore \quad \frac{p_s}{E_s} = \frac{p_b}{E_b} \text{ or } p_s = \frac{E_s}{E_b} \times p_b = \frac{30 \times 10^6}{16 \times 10^6} \times p_b = \frac{15}{8} p_b.$$

Eliminating p_s we get, $\cdot 1963 \times \frac{23}{8} \times p_b = 2000$

$\therefore \quad p_b = 3544$ lb/in^2.

and $\quad p_s = 6645$ lb/in^2.

$\therefore \quad P_b = \cdot 1963 \times 3544 = 695{\cdot}8$ lb-wt.

and $\quad P_s = \cdot 1963 \times 6645 = 1304{\cdot}2$ lb-wt.

As the cross-piece is in equilibrium under the load P and the pulls P_s and P_b of the rods, taking moments about the lower end of the steel rod,

$P_b \times 30 = P \times x$

$$\text{or} \quad x = \frac{30 \times 695{\cdot}8}{2000} = 10{\cdot}44 \text{ inches.}$$

11. Stresses due to changes of temperature: When a material undergoes natural expansion or contraction as a result of a rise or drop in temperature, there is no strain and therefore no resistance or stress. It is only when the natural expansion or contraction due to a change in temperature is prevented, that external forces will be called into play and stresses and strains will be set up. For instance, if a bar of length l and sectional area A is heated through a range of t

degrees but is prevented by end-grips from expanding, external forces P will be exerted by the end-grips causing compressive strain and the bar will have to offer compressive stresses to resist the forces P.

The natural length of the bar after heating should be $l_1 = l\ (1 + \alpha t)$ where α is the co-efficient of expansion of the material. Since the end-grips prevent the expansion δl which is $(l_1 - l)$ or $l.\alpha.t$, the compressive strain imposed by the grips is $e = \frac{\delta l}{l} = \alpha.t$. The compressive stress set up will, therefore, be $p = E.e = E.\alpha.t$ and the push P exerted by the grips will be $P = p.A = E.A.\alpha.t$. If the grips yield by an amount δ, the expansion prevented is $(l.\alpha.t - \delta)$ and the imposed strain $e = \frac{(l.\alpha.t - \delta)}{l} = (\alpha.t - \frac{\delta}{l})$ and the induced stress $p = E\ (\alpha.t - \frac{\delta}{l})$.

Similarly, if a bar is heated and then its ends are held firmly by grips so as to prevent its return to its original length on cooling, tensile strain will have been imposed on it by the pulls exerted by the end-grips and tensile stresses will be induced in the bar.

Example 12.

A steel bar $\frac{3}{4}''$ *in diameter and 15 feet long is heated through 100°F above the atmospheric temperature when the ends are clamped. On cooling back to atmospheric temperature, it is found that the clamps have yielded by ·02″. Find the pull exerted on the bar by the end clamps.* $E = 13000$ *tons/in*^{2}. $\alpha = {\cdot}0000062$ *per °F.*

Tensile strain imposed by the end-grips,

$$e = \alpha t - \frac{\delta}{l} = {\cdot}0000062 \quad 100 - \frac{{\cdot}02}{180}$$

$$= {\cdot}00062 - {\cdot}00011$$

$$= {\cdot}00051$$

$$p = E.e = 13000 \times {\cdot}00051 = 6{\cdot}63 \text{ tons/in}^2.$$

$$P = p.A = 6{\cdot}63 \times {\cdot}44 = 2{\cdot}92 \text{ tons.}$$

Example 13.

A mild steel bar 1″ in diameter and 12″ long is encased in a brass tube whose external diameter is $1\frac{1}{2}$″ and internal diameter is 1″. The composite bar is heated through 100°F. Calculate the stresses induced in each metal. α for steel = ·0000062; α for brass = ·000010 per °F. E for steel = 30×10^6 lb./in^2 and E for brass = 14×10^6 lb./in^2.

Consider a unit length of the composite bar and let e be its elongation under the rise of temperature. The steel bar will be pulled by the brass casing and will stretch more than its natural expansion, thus suffering a tensile strain. The brass tube will not be able to undergo its full natural expansion being restrained by the steel rod with its lesser coefficient of expansion. The brass tube will therefore be under compressive strain. Let e_s be the tensile strain of the steel rod and e_b the compressive strain of the brass tube.

Then $$e_s = e - \alpha_s . t$$

and $$e_b = \alpha_b . t - e$$

adding, $$e_s + e_b = t\ (\alpha_b - \alpha_s)$$

or $$\frac{p_s}{E_s} + \frac{p_b}{E_b} = t\ (\alpha_b - \alpha_s)$$

Substituting,

$$\frac{p_s}{30 \times 10^6} + \frac{p_b}{14 \times 10^6} = 100\ (\cdot 000010 - \cdot 0000062)$$

$$= 100 \times \cdot 0000038$$

$$= \cdot 00038.$$

$\therefore$ $$p_s + \frac{30}{14} p_b = \cdot 00038 \times 30 \times 10^6$$

$$= 11400$$

or $$p_s + 2{\cdot}14 p_b = 11400 \ldots\ldots\ldots\ldots (1)$$

Also, Pull on steel = Push on brass

$\therefore$ $$p_s A_s = p_b A_b$$

or $$p_s \times \frac{\pi}{4} \times 1 = p_b \times \frac{\pi}{4}\ (1{\cdot}5^2 - 1^2)$$

$\therefore$ $$p_s = 1{\cdot}25\ p_b \ldots\ldots\ldots\ldots (2)$$

Substituting in (1),

$$p_b \times 3{\cdot}39 = 11400$$

$$\therefore \qquad p_b = 3360 \text{ lb./in}^2.$$

$$\text{and} \qquad p_s = 4200 \text{ lb./in}^2.$$

12. Shrinking-on: When a thin tyre is shrunk on to a wheel of diameter D, its internal diameter d is a little less than the wheel diameter D. When the tyre is heated through the necessary range, its circumference πd will increase to πD. In this condition, it will be slipped on to the wheel. When it cools, it wants to return to its original length πd but the wheel, if it is assumed to be rigid, prevents it from doing so. The tensile strain e thus imposed on it is $\dfrac{\pi D - \pi d}{\pi d}$ or $\dfrac{D - d}{d}$. This is known as circumferential or hoop strain. The accompanying circumferential or hoop stress $p = E.e = \dfrac{E(D-d)}{d}$.

Example 14.

A thin tyre is shrunk on to a wheel of 4 feet diameter. If the wheel is assumed to be rigid, calculate what the internal diameter of the tyre should be, if after shrinking on, the hoop stress in the tyre is 6 tons/in^2. α *for the tyre* $= {\cdot}0000062$ *per* $^\circ F$*; and* $E = 13000$ *tons/in*^{2}*. Find the least temperature to which the tyre must be heated above that of the wheel before it could be slipped on.*

$$\text{Hoop stress } p = \frac{E\,(D-d)}{d}$$

$$\therefore \qquad 6 = \frac{13000\,(D-d)}{d}$$

$$\therefore \qquad \frac{D-d}{d} = \frac{6}{13000}$$

$$\text{or} \qquad \frac{D}{d} = \left(1 + \frac{6}{13000}\right)$$

$$\therefore \qquad \frac{d}{D} = \left(1 + \frac{6}{13000}\right)^{-1}$$

$$= 1 - \frac{6}{13000}, \text{ approximately}$$

$$\therefore \quad d = 48 - \cdot 022 = 47 \cdot 978''.$$

Also $$\pi D = \pi d\ (1 + \alpha t)$$

$$\therefore \quad \alpha . t = \frac{D-d}{d} = \frac{6}{13000}$$

$$\therefore \quad t = \frac{6 \times 10^7}{13000 \times 62} = 74 \cdot 4\ ^\circ\text{F}.$$

13. Stresses on inclined sections of a bar under tension or compression: Consider a bar of uniform sectional area A under an axial pull P. The intensity of stress on the normal cross-section AB is $p = \frac{P}{A}$. The intensity of stress set up by the section CD inclined at θ° to the normal cross-section AB is obviously P divided by the area of the resisting section which is $A\sec\theta$. The intensity of stress $= \frac{P}{A\sec\theta}$ $= \frac{P}{A}\cos\theta = p.\cos\theta$.

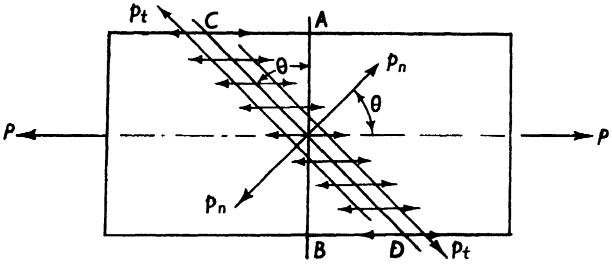


FIG. 8

This is naturally less than p because the resisting section has a bigger area. But it is neither normal nor tangential to the section CD. As failure can occur either (i) by tension or compression or (ii) by shear, it is very necessary to know the normal and tangential stresses put up by any resisting section. Resolving the pull P, normally and tangentially to the section CD, we get $P_n = P.\cos\theta$ and $P_t = P.\sin\theta$. The normal and tangential stress intensities of the section are therefore given by:

$$p_n = \frac{P_n}{A.\sec\theta} = \frac{P\cos\theta}{A.\sec\theta} = p.\cos^2\theta \text{ and}$$

$$p_t = \frac{P_t}{A.\sec\theta} = \frac{P\sin\theta}{A.\sec\theta} = p.\sin\theta.\cos\theta.$$

This could have been obtained also by considering a unit area of the section *CD*. The stress on it is $p.\cos\theta$ in the direction of *P*. Resolving it normally and tangentially

$$p_n = p.\cos\theta.\cos\theta = p\cos^2\theta$$

$$p_t = p.\cos\theta \times \sin\theta \text{ or } p\sin\theta.\cos\theta, \text{ as before.}$$

As failure will occur either by the bar being pulled to pieces (or being crushed if there is compression) or by shearing off, it is important to locate the plane or planes carrying the largest direct stress and those carrying the largest shear. For maximum direct stress, p_n must be a maximum. This is obviously p when $\cos^2\theta = 1$ i.e. $\theta = 0$. Thus, the normal cross-section carries the largest direct stress. On any other section, p_n is less than p.

For planes of maximum shear, $p_t = p \sin\theta.\cos\theta$ must be a maximum. p_t may be expressed as $\frac{1}{2}p.\sin 2\theta$. The maximum value of p_t is $\frac{1}{2}p$, when $\sin 2\theta = 1$ i.e. when $2\theta = 90°$ or $270°$. The planes offering the maximum shear stress are thus two, inclined at 45° and 135° to the normal section.

It may be noted that they are at right angles to each other and inclined at 45° to the plane carrying the maximum direct stress.

Example 15.

The normal stress intensity on the cross-section of a tie bar is 8 tons/in^2. Calculate the normal, tangential and resultant stress intensities on a section inclined at 35° to the normal section.

Resultant stress intensity $= p.\cos\theta$
$= 8 \times \cdot 8192 = 6{\cdot}55$ tons/in^2.
$p_n = p\cos^2\theta = 6{\cdot}55 \times \cdot 8192 = 5{\cdot}37$ tons/in^2.
$p_t = p.\cos\theta\sin\theta = 6{\cdot}55 \times \cdot 5736 = 3{\cdot}76$ tons/in^2.

The maximum possible shear on any plane is $\frac{1}{2} \times p = 4$ tons/in^2; and these planes are inclined at 45° to the normal section.

14. *Simple Shear:* *Next, let us consider a section which* is subjected to forces P tangentially or across the section which puts up a shear stress of intensity $q = \frac{P}{A}$, A being the area of resisting section.

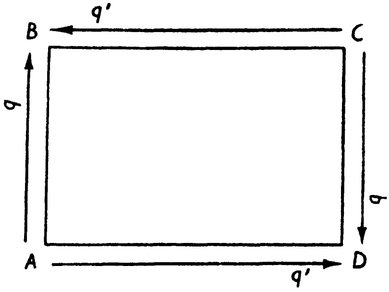


FIG. 9

Consider an elementary rectangular block $ABCD$ of thickness l with forces P acting across the faces AB and CD which will offer shear stress q given by $P = q.AB.l$ or $q.CD.l$. These forces being equal and opposite cannot balance but constitute a couple whose moment is $P.BC$ or $q.AB.BC.l$. Now a couple cannot be balanced except by a couple of the same magnitude and opposite in direction. On the faces BC and AD of the block, there must therefore be opposite forces P' inducing a stress intensity q' given by $P' = q'.BC.l$ or $q'.AD.l$. They constitute the balancing couple whose moment is $P'.AB$ or $q'.AB.BC.l$.

Equating the two couples,

$$q.AB.BC.l = q'.AB.BC.l$$

or $$q = q'.$$

Thus, we get the important result that a state of shear across a plane cannot exist unless it is accompanied by a balancing shear of the same intensity across a plane at right angles to it. This is called the complementary shear and the two together constitute a state of simple shear.

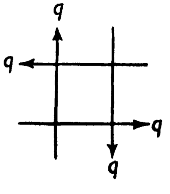


FIG. 10

As soon as we note a shear stress q at a point in a strained material, we must, at the same time, mark the complementary shear stress q at right angles as in fig. 10.

15. State of simple shear: Stresses on inclined sections: Consider an elementary rectangular block *ABCD* of thickness *l*, having shear intensities *q* across the faces *AB*, *CD* and *AD*, *BC*. Let us find the normal and tangential stress intensities on a plane *CE* inclined at θ° to the face *CD*. The force P_1 on the face *CD* is $q.CD.l \downarrow$. The force P_2 on *ED* is $q.ED.l \rightarrow$. Resolving these forces normally and tangentially to *CE*, we get,

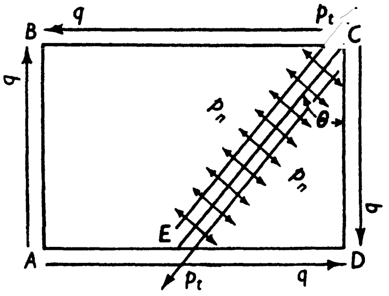


FIG. 11

$$P_n = P_1 \sin\theta + P_2 \cos\theta$$
$$= q.CD.l \sin\theta + q.ED.l\cos\theta$$

and
$$P_t = P_1 \cos\theta - P_2 \sin\theta$$
$$q.CD.l\cos\theta - q.ED.l\sin\theta.$$

The normal and tangential intensities of stress are therefore given by:

$$p_n = \frac{P_n}{CE.l} = \frac{q.CD.l \sin\theta + q.ED.l\cos\theta}{CE.l}$$
$$= 2q \sin\theta\cos\theta$$
$$= q.\sin 2\theta$$

and
$$p_t = \frac{P_t}{CE.l} = \frac{q.CD.l \cos\theta - q.ED.l. \sin\theta}{CE.l}$$
$$= q \,(\cos^2\theta - \sin^2\theta) = q.\cos 2\theta.$$

For the planes carrying the maximum direct stress, p_n must be a maximum. The maximum value of p_n is obviously $\pm q$ when $\sin 2\theta = \pm 1$ or $2\theta = 90°$ or $-90°$ i.e. when $\theta = \pm 45°$. The planes are thus mutually at right angles and inclined at 45° to the planes of simple shear, both having the *maximum direct stress q which, however, is tensile for one plane* and compressive for the other. It will be noted that these planes carry no shear stress since $p_t = q \cos 2\theta$ is zero when $\theta = \pm 45°$.

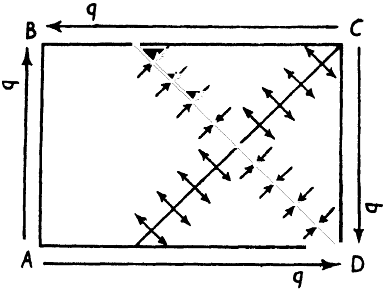


FIG. 12

As for the planes of maximum shear, the maximum value of $p_t = q.\cos 2\theta$ is q when $\cos 2\theta = 1$ i.e. $2\theta = 0°$ or $180°$, or, when $\theta = 0°$ or $90°$. The planes of simple shear are thus the planes carrying the maximum shear stress. On any other plane the tangential or shear stress p_t is less than q.

It will be noted that these planes are mutually at right angles and inclined at 45° to the planes carrying direct stress only.

We thus get the important deduction that a state of simple shear produces tensile and compressive direct stresses across mutually perpendicular planes inclined at 45° to those of pure shear and that the intensity of either of the direct stresses is the same as the intensity of the pure shear stress.

This could have been derived simply by studying a square block $ABCD$ of thickness l, instead of the rectangular one, its faces carrying a shear stress of intensity q. Consider the diagonal plane AC. The force P_1 on the face CD is $q.CD.l. \downarrow$. The force P_2 on the face AD is $q.AD.l. \rightarrow$. $P_1 = P_2$ since $AD = CD$, the block being square in section. Resolving normally to the plane AC, $P_n = P_1 \cos 45° + P_2 \sin 45° = 2P_1 \cos 45° = \sqrt{2}P_1$ and $P_t = P_1 \sin 45° - P_2 \cos 45° = 0$.

$$p_n = \frac{P_n}{AC.l} = \frac{\sqrt{2}.q.CD.l}{\sqrt{2}.CD.l} = q \text{ and is tensile.}$$

$$p_t = \frac{P_t}{AC.l} = 0.$$

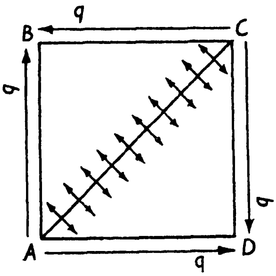


FIG. 13

The diagonal plane AC is subjected to a direct tensile stress of intensity q. Similarly it can be proved that the diagonal plane BD carries a direct compressive stress of the same intensity.

16. Linear strain of the diagonal BD: Let ϕ be the shear strain accompanying the state of simple shear q. The section $ABCD$ will be distorted to AB_1C_1D through the angle ϕ assuming AD to be fixed.

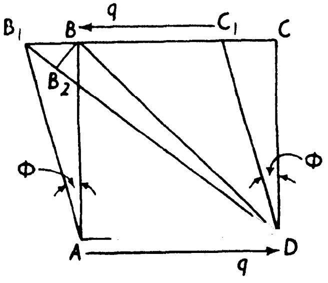


FIG. 14

$$\text{Linear strain of } BD = \frac{DB_1 - DB}{DB} = \frac{B_1B_2}{DB} = \frac{BB_1 \cos 45^\circ}{\sqrt{2}.AB}$$

$$= \frac{1}{2}\frac{BB_1}{AB} = \frac{1}{2}\phi \text{ or half the shear strain}$$

and is tensile. Similarly the linear strain of the diagonal AC is also $\frac{1}{2}\phi$ but is compressive.

17. Relation between the Moduli of Elasticity and Rigidity for a given material: The linear strain of the diagonal BD is seen to be $\frac{1}{2} \times \phi$ or $\frac{1}{2} \times \frac{q}{N}$. We could have derived its strain differently from considerations of the direct tensile stress q across the plane AC, accompanied by the direct compressive stress q across the plane BD. The effect of the former is a tensile strain $\frac{q}{E}$ on all lengths perpendicular to the plane AC. The diagonal BD will therefore undergo a tensile strain of $\frac{q}{E}$ due to this stress. The effect of the compressive stress q on the plane BD will be a compressive strain $\frac{q}{E}$ in its own direction and a lateral tensile strain of $\frac{1}{m} \quad \frac{q}{E}$ on all lengths at right angles. The effect of this stress on the diagonal BD will therefore be $\frac{1}{m} \times \frac{q}{E}$. The combined effect of the two direct stresses on the diagonal will, therefore, be a tensile strain of $\frac{q}{E} + \frac{1}{m} \quad \frac{q}{E}$ or $\frac{q}{m}\ (1 + \frac{1}{m})$. But this linear strain has already been found to be $\frac{1}{2} \cdot \frac{q}{N}$.

$$\text{Equating,} \quad \frac{q}{2N} = \frac{q}{E}\ (1 + \frac{1}{m})$$

$$\text{or} \qquad E = 2N\ (1 + \frac{1}{m})$$

18. Bulk Modulus: When a body is subjected to three mutually perpendicular like direct stresses of the same intensity, the ratio of this direct stress and the corresponding volumetric strain of the body is defined as the Bulk Modulus of the body. It is usually denoted by the letter K.

Consider a cube $ABCDA_1B_1C_1D_1$ of side l with its faces subjected to direct stresses of intensity p. Consider the deformation of one of the edges, say BC, due to these three pairs of stresses.

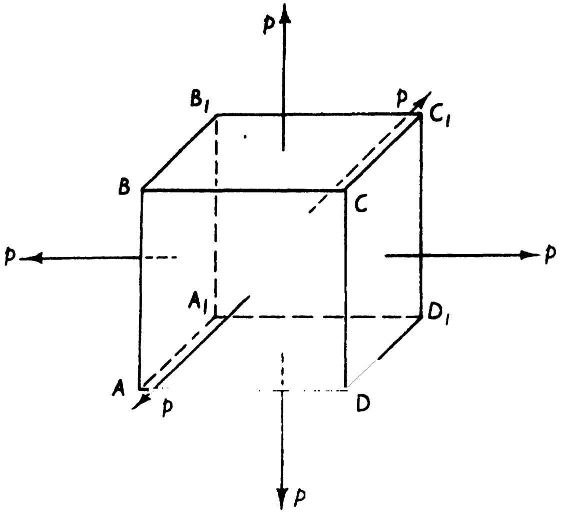


FIG. 15

Due to the stresses p on the faces CC_1D_1D and BB_1A_1A, BC suffers a tensile strain of $\frac{p}{E}$; due to the pair of stresses p on the faces $BB_1\,C_1C$ and AA_1D_1D, there will be a tensile strain of $\frac{p}{E}$ in its own direction but a compressive lateral strain of $\frac{1}{m} \times \frac{p}{E}$ in all directions at right angles. BC therefore suffers a compressive strain of $\frac{1}{m} \times \frac{p}{E}$. Similarly, due to the third pair of stresses p on the faces $ABCD$, and $A_1B_1C_1D_1$, BC suffers a compressive strain of $\frac{1}{m} \times \frac{p}{E}$.

The total tensile strain of BC due to the three pairs of stresses is given by:

$$\frac{\delta l}{l} = \frac{p}{E} - \frac{1}{m} \times \frac{p}{E} - \frac{1}{m} \times \frac{p}{E} = \frac{p}{E}\left(1 - \frac{2}{m}\right).$$

The original volume of the cube

$$V = l^3$$

$$\therefore \quad \delta V = 3l^2 . \delta l$$

$$\text{and } \frac{\delta V}{V} = 3 . \frac{\delta l}{l} = 3 \frac{p}{E} (1 - \frac{2}{m})$$

By definition,

$$\frac{p}{\text{volumetric strain}} = K$$

$$\therefore \quad \frac{p}{\frac{3p}{E}(1 - \frac{2}{m})} = K$$

$$\text{or} \qquad E = 3K (1 - \frac{2}{m})$$

The relation $E = 2N (1 + \frac{1}{m}) = 3K (1 - \frac{2}{m})$ between the three moduli for a given material is thus established.

Example 16.

A bar of steel $1\frac{1}{8}''$ *in diameter was subjected to a tensile load of 6 tons and the measured extension on an 8″ gauge length was ·0036″ and the change in diameter was ·00015″. Calculate Poisson's Ratio and the values of the three moduli.*

Area of section = ·994 in^2.

$$\text{Stress } p = \frac{P}{A} = \frac{6}{\cdot 994} = 6 \text{ tons/in}^2.$$

$$\text{Strain } e = \frac{\delta l}{l} = \frac{\cdot 0036}{8} = \cdot 00045$$

$$E = \frac{p}{e} = \frac{6}{\cdot 00045} = 13300 \text{ tons/in}^2.$$

$$\text{Lateral strain } \frac{\delta d}{d} = \frac{\cdot 000150}{1 \cdot 125} = \cdot 000133.$$

$$\text{But lateral strain} = \frac{1}{m} \times \text{linear strain}$$

$$\therefore \qquad \cdot 000133 = \frac{1}{m} \times \cdot 00045$$

$\therefore$ Poisson's ratio $\frac{1}{m} = \cdot 2955.$

Also $E = 2N\left(1 + \frac{1}{m}\right) = 3K\left(1 - \frac{2}{m}\right)$

$= 2N \times 1\cdot 2955 = 3K \times \cdot 4090$

$\therefore \quad N = \frac{13300}{2\cdot 591} = 5145$ tons/in².

and $K = \frac{13300}{1\cdot 227} = 10860$ tons/in².

Example 17.

The modulus of Rigidity for a material is 2640 tons/in². A $\frac{3}{8}''$ *diameter rod of the material was subjected to an axial pull of* $\frac{1}{2}$ *ton and the change in diameter was observed to be 0·000078″. Calculate Poisson's ratio and the modulus of Elasticity for the material.*

Area of section = ·1104 in².

$$\text{Stress } p = \frac{P}{A} = \frac{\cdot 5}{\cdot 1104} = 4\cdot 56 \text{ tons/in}^2.$$

Lateral strain $\frac{\delta d}{d} = \frac{\cdot 000078}{\cdot 375} = \cdot 000208.$

But lateral strain $= \frac{1}{m} \times \frac{p}{E}$

$\therefore \quad \frac{p}{mE} = \cdot 000208$

and $\quad mE = \frac{4\cdot 56}{\cdot 000208} = 21930.$

Also $\quad E = 2N\left(1 + \frac{1}{m}\right)$

$\therefore \quad mE = 2N(m + 1)$

or $\quad 21930 = 2 \times 2640\,(m + 1)$

$\therefore \quad m + 1 = 4\cdot 153$

and $\quad m = 3\cdot 153$

$\therefore \quad E = \frac{21930}{3\cdot 153} = 6950$ ton/in².

Poisson's ratio $\frac{1}{m} = \frac{1}{3\cdot 153} = \cdot 317.$

EXAMPLES I

(1) A wooden tie 20 feet long, 3″ wide and 4″ thick is subjected to an axial pull of 10000 lb-wt. and the stretch is ·154″. Calculate the value of E for the timber. ($1{\cdot}3 \times 10^6$ lb./in^2.)

(2) The rod of a hydraulic lift is 40 feet long and $1\frac{1}{4}$″ in diameter. It is attached to a plunger $4\frac{1}{2}$″ in diameter working under a pressure of 1200 lb/in^2. If $E = 30 \times 10^6$ lb/in^2., find the alteration in the length of the rod. (·249″.)

(3) A tie-bar 1″ in diameter carries a load which causes a stress of 16000 lb/in^2. It is attached to a cast iron bracket by means of four bolts which can be stressed to 12000 lb/in^2. Find the diameter of the bolts. (·577″ say $\frac{5}{8}$″.)

(4) A steel punch can be worked to a compressive stress of 50 tons/in^2. Find the least diameter of hole which can be punched through a steel plate $\frac{1}{2}$″ thick if its ultimate shear strength is 20 tons/in^2. (·8″.)

(5) A short strut is made up of two lengths of standard Tee-sections 6″ × 3″ × $\frac{1}{2}$″ riveted back to back. If it carries a load of 50 tons, find the compressive stress. If the ultimate crushing stress for the material is 30 tons/in^2., what is the factor of safety? (5·9 tons/in^2.; 5·1.)

(6) The ultimate tensile stress of mild steel is 30 tons/in^2. A tie-bar of equal-angle section has to carry an axial pull of 18 tons. Give suitable dimensions for the section if its mean thickness is $\frac{3}{8}$″. Use a factor of safety of 4. ($3\frac{1}{2}$″ × $3\frac{1}{2}$″ × $\frac{3}{8}$″.)

(7) A steel bar 2″ × 2″ in section, 10 feet long, is subjected to an axial pull of 20 tons. Taking $E = 13000$ tons/in^2. and Poisson's ratio as ·3, calculate the alterations in length, side and volume of the bar. Calculate also the amount of energy stored in the bar during the extension. (·0468″; ·00024″; ·075 cub-inch; ·468 inch-ton.)

(8) A mild steel flat 6″ wide by $\frac{3}{4}$″ thick, 20 feet long, carries an axial pull of 30 tons. If $E = 13000$ tons/in^2., and $\frac{1}{m} = {\cdot}26$, calculate the change in length, width, thickness and volume of the flat.
(·122″; ·0008″; ·0001″; ·265 cub. inch.)

(9) A bar of steel 10 feet long is subjected to a pull of 8 tons. It is $1\frac{1}{4}$″ in diameter for 3 feet of its length, $1\frac{1}{8}$″ in diameter for 4 feet and 1″ in diameter for the remaining 3 feet. Find the total elongation of the bar and the energy stored in it. $E = 13000$ tons/in^2.
(·076″; ·3 inch-ton.)

(10) A bar of steel is 20″ long. The two ends are respectively $1\frac{1}{4}$″ and $1\frac{1}{8}$″ in diameter and each is 5″ long, the middle portion being

1″ in diameter. Determine the contraction in length if the bar carries an axial thrust which produces a stress of 6 tons/in^2., in the smallest section. $E = 13500$ tons/in^2. (·00762″.)

(11) A straight bar of steel is 10 feet long and has a rectangular section which varies uniformly from 4″ × ½″ at one end to 1″ × ½″ at the other. What change will occur in its length when subjected to an axial load of 5000 lb.? $E = 30 \times 10^6$ lb./in^2. (·0185″.)

(12) A steel rod circular in section tapers uniformly from 1″ diameter at one end to ¼″ diameter at the other in a length of 30 inches. How much will it stretch under an axial pull of ½ ton? $E = 13000$ tons/in^2. (·0059″.)

(13) A reinforced concrete column of square section has to be designed to carry an axial load of 35 tons, the reinforcement consisting of vertical rods, one at each corner. The sectional area of reinforcement may be 3% of the sectional area of the column. If the permissible compressive stress in concrete is 600 lb/in^2., and the modular ratio of steel and concrete is 15, determine the dimensions of the column and the diameter of the reinforcing rods. (10″ × 10″; 1″.)

(14) A copper bar 1½″ in diameter is rigidly attached at both ends to the inside of a steel tube which is 2″ in external diameter, thickness of metal ¼″. The composite section is then subjected to an axial pull of 10 tons. Find the stress induced in each metal and the extension on a length of 4 feet. $E_c = 7000$ tons/in^2. $E_s = 13000$ tons/in^2.
(2·3 tons/in^2.; 4·3 tons/in^2.; ·016″.)

(15) Three vertical rods, equal in length, each ½″ in diameter, are equispaced in a vertical plane and jointly support a load of 1·18 tons, the rods being so adjusted as to share the load equally. If a further load of 1·32 tons is added, find the stress in each rod, the middle one being copper and the two outer ones being steel.
$E_c = 7000$ tons/in^2.; $E_s = 13000$ tons/in^2.
(3·43 tons/in^2. for copper; 4·66 tons/in^2. for steel.)

(16) A steel tube 3½″ internal diameter and ¼″ thick has an inner and an outer brass liner of the same length and thickness. The composite section carries an axial thrust of 15 tons. Calculate the load carried by each tube and the amount it shortens. Length 6″. $E_s = 30 \times 10^6$ lb/in^2.; $E_b = 14 \times 10^6$ lb/in^2.
(Inner tube 3·13 tons; central 7·77 tons; outer 4·1 tons.)

(17) A rolled steel joist is 20 feet long at 60°F. Calculate its elongation when the temperature is 110°F. If this elongation is prevented, calculate the stress induced in the section and the thrust exerted if the area of section is 9·45 in^2. α for steel = ·0000062 per °F and $E = 13000$ tons/in^2. (4·03 tons/in^2.; 38·1 tons.)

(18) Two walls, 20 feet apart are stayed together by a steel rod 1¼″ in diameter by metal plates and nuts screwed at each end The

nuts are screwed when the bar is heated to a temperature of 260°F. Calculate the pull exerted by the bar on cooling to atmospheric temperature of 60°F if the ends do not yield, also if they yield by $\frac{1}{8}$ inch. α for steel = ·0000062 per °F and E = 13000 tons/in^2.
(19·83 tons; 11·52 tons.)

(19) A copper rod $1\frac{1}{2}''$ diameter is enclosed within a steel tube $1\frac{3}{4}''$ internal diameter and $\frac{1}{8}''$ thick, the ends being rigidly connected together. The combination is then heated through 80°F. Find the intensity of stress in each metal. α for steel = ·0000062 per °F and for copper ·00001 per °F. E_s = 13000 tons/in^2. and E_c = 7000 tons/in^2.
(p_c = ·93 tons/in^2.; p_s = 2·23 tons/in^2.)

(20) In the above example, if a pin $\frac{1}{2}''$ in diameter is driven transversely to the axis of the bar at each end to secure the ends rigidly, calculate the intensity of shear stress that would be induced in the section of the pin for the 80° rise of temperature. (4·19 tons/in^2.).

(21) In a tensile test on a steel tube of external diameter $\frac{3}{4}''$, $\frac{1}{2}''$ bore, an axial load of ·2 ton produced an elongation of ·00018″ in a length of 3″, while the outer diameter suffered a compression of ·000013″. Calculate the values of Poisson's ratio, E, N and K.
(·2883; 13600 tons/in^2.; 5280 tons/in^2.; 10700 tons/in^2.).

(22) A load of 2·5 tons rests on three vertical short struts, each $\frac{1}{2}''$ in diameter and equispaced in a vertical plane. The central pillar is copper and the two outer ones are steel. The pillars are so adjusted that at atmospheric temperature, each carries one-third the total load. The temperature is then raised through 50°F. Calculate the stress in each pillar before and after the rise of temperature. E_c=7000 tons/in^2.; E_s = 13000 tons/in^2; α_c = ·00001 per °F; α_s = ·0000062 per °F.
(4·24 tons/in^2.; 5·28 tons/in^2. in copper, 3·72 tons/in^2. in steel.)

(23) A 2″ diameter round bar, 20 feet long, carries a load of 22 tons and stretches ·13″ under the load. Estimate the value of E and the maximum shear stress induced in the bar.
(12900 tons/in^2; 3·5 tons/in^2.)

(24) In the above example, calculate the normal and tangential stress intensities across a plane inclined at 25° to the normal section of the bar. (5·75; 2·68 tons/in^2.)

(25) A 6″ × $\frac{3}{4}''$ mild steel flat carries an axial pull of 30 tons. Calculate the normal and tangential stress-intensities across planes making angles of 30°, 45° and 60° with the axis of the flat.

$$\left\{\begin{array}{llll} 30°; & p_n = 1{\cdot}67 \text{ tons/in}^2., & p_t = 2{\cdot}89 \text{ tons/in}^2. \\ 45°; & p_n = 3{\cdot}33 \text{ tons/in}^2., & p_t = 3{\cdot}33 \text{ tons/in}^2. \\ 60°; & p_n = 5, \quad \text{tons/in}^2., & p_t = 2{\cdot}89 \text{ tons/in}^2. \end{array}\right\}$$

(26) A compound bar consists of a central steel strip 1" wide $\times \frac{1}{4}$" thick placed between two strips of brass each 1" wide and t inch thick. The strips are firmly fixed together, to form a compound bar of rectangular section 1" wide and $(\frac{1}{4} + 2t)$ inch thick. Determine (a) the thickness t of the brass strips which will make the apparent modulus of elasticity of the compound bar 10000 tons/in^2. and (b) the maximum axial pull the bar can then carry if the stress is not to exceed 10 tons/in^2, in either the brass or the steel. Take the values of E for steel and brass as 13200 tons/in^2. and 7260 tons/in^2. respectively. (L. U.)

($t = 0{\cdot}146$"; 4·106 tons.)

(27) A round steel bar, $1\frac{1}{8}$" diameter and 15" long, is placed concentrically within a brass tube which has an outside diameter of $1\frac{5}{8}$" and inside diameter of $1\frac{1}{4}$"; the length of the tube exceeds that of the bar by 0·005 inch. Rigid plates are placed on the ends of the tube through which an axial compressive force is applied to the compound bar. Determine the compressive stresses in the bar and tube due to a force of 6 tons. E for steel = 13200 tons/in^2. E for brass = 6500 tons/in^2. (L. U.)

($p_s = 2{\cdot}96$ tons/in^2.; $p_b = 3{\cdot}63$ tons/in^2.)

(28) A mild steel rod ABC of circular section transmits an axial pull. The total length is 54", AB being 30" long and BC 24" long. AB is $\frac{7}{8}$" diameter (sectional area 0·6 in^2.) and BC is $\frac{3}{4}$" diameter (sectional area 0·44 in^2.). If the total change in length is 0·0282 inch, determine for the separate parts AB and BC the changes in (a) length, (b) diameter and (c) volume. Take Poisson's ratio = 0·3. (L. U.)

{ (a) ·01349"; ·01471"; (b) ·000118"; ·000138";
(c) ·00323 cub. in.; ·00257 cub. in. }

CHAPTER II

PRINCIPAL PLANES AND PRINCIPAL STRESSES

1. Introductory: We have studied in detail a state of direct stress—tension or compression—and the state of simple shear. In each case, we can work out the normal and tangential intensities of stress on any plane. But the planes to pick out are necessarily the planes carrying the maximum direct stress and those carrying the maximum shear, because failure will occur either by a section (i) tearing off, (ii) being crushed, or (iii) shearing off.

In the simple cases studied so far we have noted that the planes of maximum direct stress have no tangential stress across them. We have also noted that planes of maximum shear are mutually at right angles and inclined at 45° to those carrying direct stress only.

2. Principal planes: It may be stated, in general, that at any point in a strained material, there are three mutually perpendicular planes which carry direct stresses only and *no* tangential stress whatever. Of these three direct stresses, one is a maximum and one a minimum. These particular planes which have no shear stress on them are known as *Principal Planes* and the direct stresses across them are known as *Principal Stresses*. In the study of stresses and strains at any point in a strained material, it is very necessary to locate the principal planes, because one of them carries the largest direct stress—tensile or compressive. It is necessary for another reason also because planes of maximum shear stress are inclined at 45° to the principal planes. Once the principal planes and the principal stresses have been picked out, the critical planes carrying the largest direct stress and those carrying the maximum shear have been located. In most of the problems that we have to deal with, one of the principal stresses is zero. We are therefore left with two principal stresses, one of which will be a maximum and the other a minimum. The direct stress p_n across any other plane will have some intermediate value between p_1 the maximum and p_2 the minimum principal stress.

3. Two Principal Stresses: To prove the statement made above, let p_1 and p_2 be the principal stresses, both like, across the principal planes *AB* and *BC* in fig. 16. Let us consider a plane *AC* inclined at θ° to the principal plane *AB* and find the normal and tangential stress intensities across it.

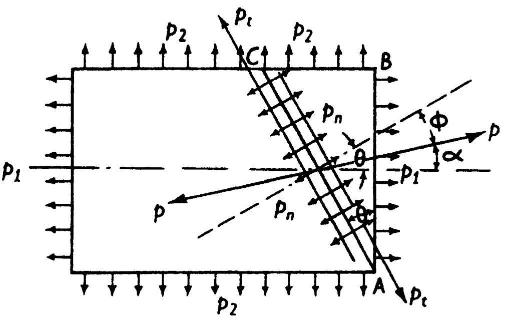


FIG. 16

The force P_1 on the plane *AB* (supposed vertical) is equal to $p_1.AB.l$ where *l* is the thickness of the piece *ABC*. The force P_2 on the face *BC* is $p_2.BC.l$. Resolving P_1 and P_2 normally and tangentially to *AC*, we get,

$$P_n = P_1\cos\theta + P_2\sin\theta = p_1.AB.l\cos\theta + p_2.BC.l\sin\theta$$

and $P_t = P_1\sin\theta - P_2\cos\theta = p_1.AB.l.\sin\theta - p_2.BC.l\cos\theta.$

The normal and tangential stress intensities set up by *AC* are therefore given by,

$$p_n = \frac{P_n}{AC.l} = \frac{p_1.AB.l\cos\theta + p_2.BC.l\sin\theta}{AC.l}$$

$$= p_1\cos^2\theta + p_2\sin^2\theta$$

and $$p_t = \frac{P_t}{AC.l} = \frac{p_1.AB.l\sin\theta - p_2.BC.l\cos\theta}{AC.l}$$

$$= (p_1 - p_2)\ \sin\theta.\cos\theta.$$

We could have obtained this result, simply, by considering a unit area of the inclined plane *AC*. The effect of p_1 on this area is $p_x = p_1\cos\theta$ in the direction of p_1 and that of p_2 on this area is $p_y = p_2 \sin\theta$ in the direction of p_2. Resolving normally and tangentially to the unit area, we get,

$$p_n = p_1.\cos\theta.\cos\theta + p_2\sin\theta.\sin\theta = p_1\cos^2\theta + p_2\sin^2\theta \text{ and}$$

$$p_t = p_1.\cos\theta.\sin\theta - p_2\sin\theta.\cos\theta = (p_1 - p_2).\ \sin\theta.\cos\theta.$$

The resultant intensity of stress p on AC may be looked upon as the vector sum of p_x and p_y or of p_n and p_t.

Thus, $p = \sqrt{p_x^2 + p_y^2} = \sqrt{p_n^2 + p_t^2} = \sqrt{p_1^2\cos^2\theta + p_2^2\sin^2\theta}.$

Its inclination α to the horizontal is given by,

$$\tan\alpha = \frac{p_y}{p_x} = \frac{p_2\sin\theta}{p_1\cos\theta} = \frac{p_2}{p_1}\tan\theta.$$

Its inclination ϕ to the normal is given by

$$\tan\phi = \frac{p_t}{p_n} = \frac{(p_1 - p_2)\sin\theta\cos\theta}{p_1\cos^2\theta + p_2\sin^2\theta}$$

Let us now pick out the plane of maximum direct stress and the planes of maximum shear.

For maximum direct stress, p_n must be a maximum.

$$\text{Now } p_n = p_1\cos^2\theta + p_2\sin^2\theta$$
$$= \tfrac{1}{2}p_1(1 + \cos2\theta) + \tfrac{1}{2}p_2(1 - \cos2\theta)$$
$$= \left(\frac{p_1 + p_2}{2}\right) + \left(\frac{p_1 - p_2}{2}\right)\cos2\theta.$$

This is a maximum when $\cos2\theta = +1$ i.e. when $\theta = 0$ and maximum $p_n = \left(\frac{p_1 + p_2}{2}\right) + \left(\frac{p_1 - p_2}{2}\right)$ or p_1, i.e. the plane AB carries the maximum direct stress, its value being p_1.

For the minimum value of p_n, $\cos2\theta = -1$ i.e. $2\theta = 180°$ or $\theta = 90°$. Therefore the plane BC, the second principal plane, carries the minimum direct stress and its value is $\left(\frac{p_1 + p_2}{2}\right) - \left(\frac{p_1 - p_2}{2}\right)$ or p_2.

The proposition is thus proved that one of the principal planes carries the maximum direct stress and the other the minimum. These planes being principal planes, of course carry no shear stress.

Now, for planes of maximum shear, p_t must be a maximum. The expression for p_t is $(p_1 - p_2)\sin\theta.\cos\theta$ or $\frac{1}{2}(p_1 - p_2)\sin 2\theta$. The maximum value of p_t is thus $\frac{1}{2}(p_1 - p_2)$, when $\sin 2\theta = 1$ i.e. when $2\theta = 90^\circ$ or 270°; i.e. $\theta = 45^\circ$ or 135°.

There are thus two planes offering the maximum shear stress. They are at right angles to each other and inclined at 45° to the principal planes. Across any other plane, p_t will be less than $\left(\frac{p_1 - p_2}{2}\right)$.

If the principal stress p_1 and p_2 are unlike, the normal and tangential stress intensities p_n and p_t on a plane inclined at 0° to the plane of p_1 will be given by the same expressions with the sign of the unlike stress changed.

Example 1.

The principal stresses at a point in the section of a boiler shell are 5 tons/in²., and 2½ tons/in²., both tensile. Find the normal, tangential and resultant stress intensities across a plane through the point inclined at 50° to the plane having the 5-ton stress.

$$p_n = p_1\cos^2\theta + p_2\sin^2\theta = 5 \times (\cdot 6428)^2 + 2\cdot 5(\cdot 7660)^2$$
$$= 5 \times \cdot 413 + 2\cdot 5 \times \cdot 5868 = 2\cdot 065 + 1\cdot 467 = 3\cdot 53 \text{ tons/in}^2.$$
$$p_t = (p_1 - p_2)\sin\theta\cos\theta = 2\cdot 5 \times \cdot 6428 \times \cdot 7660$$
$$= 1\cdot 23 \text{ tons/in}^2.$$
$$p^2 = (p_1\cos\theta)^2 + (p_2\sin\theta)^2 = (3\cdot 214)^2 + (1\cdot 915)^2$$
$$= 10\cdot 33 + 3\cdot 67 = 14$$
$$\therefore p = 3\cdot 74 \text{ tons/in}^2.$$
$$\tan\alpha = \frac{p_2\sin\theta}{p_1\cos\theta} = \frac{1\cdot 915}{3\cdot 214} = \cdot 5958$$
$$\therefore \alpha = 30^\circ 46'.$$

4. Graphical method: In the preceding article, we worked out the normal and tangential stress intensities across any plane inclined at θ° to the plane of the major principal stress p_1. These can also be obtained very neatly by a graphical construction as under:

Let p_1 and p_2 be the principal stresses at any point O in the strained material. With centre O, draw two concentric circles of radii OA and OB representing the stresses p_1 and p_2 to some scale. Set off the plane CD at an angle $\theta°$ with the plane of p_1. Draw the normal OEF to the plane at O, cutting the concentric circles at E and F respectively. Draw FG perpendicular to OA and EP perpendicular to FG. Join OP, which represents the resultant intensity of stress p across the plane CD. From P drop PQ perpendicular to OF. Then OQ and QP respectively represent p_n and p_t.

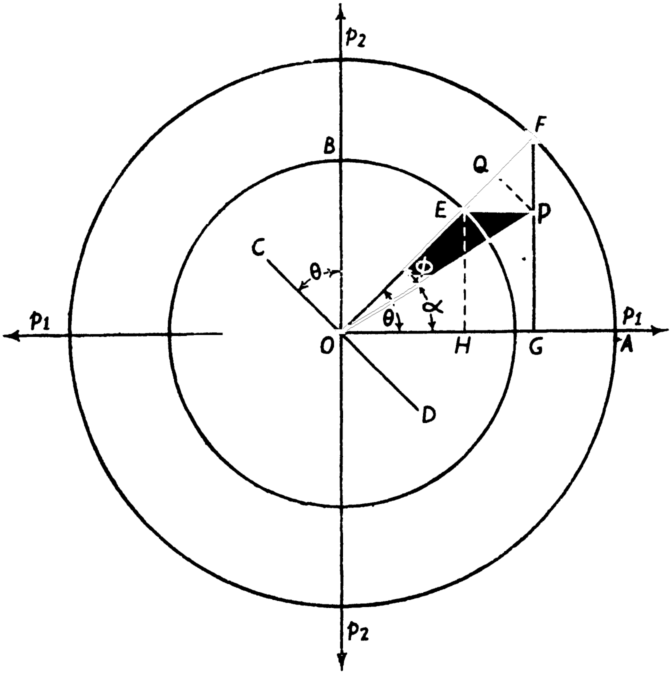


FIG. 17

Proof:

From E drop EH perpendicular to OA.

Now, $OG = OF\cos\theta = p_1\cos\theta$ or p_x of article 3.

$GP = EH = OE\sin\theta = p_2\sin\theta$ or p_y of article 3.

OP which is the vector sum of *OG* and *GP* therefore represents p.

$$OP = \sqrt{OG^2 + GP^2} = \sqrt{p_1^2 \cos^2\theta + p_2^2 \sin^2\theta}.$$

Its inclination α to *OA* is given by

$$\tan \alpha = \frac{GP}{OG} = \frac{p_2 \sin \theta}{p_1 \cos \theta} = \frac{p_2}{p_1} \tan\theta$$

$\angle POQ$ is obviously ϕ and is given by $\tan \phi = \dfrac{p_t}{p_n}$.

In the above construction, p_1 and p_2 are supposed to be like stresses. However, if they are unlike, the two components of the resultant intensity will be *OG* and *GP'*, *GP'* being equal to *GP* but opposite in sign. Produce *PG* to *P'* making $GP' = GP$. Join *OP'*. Then *OP'* will represent the resultant intensity of stress on the given plane *CD* when p_1 and p_2 are unlike stresses.

The co-ordinates of *P* referred to *OA* and *OB* as axes of reference are

$$x = OG = p_1 \cos\theta$$
$$y = GP = p_2 \sin\theta.$$

The locus of *P* as θ varies, is given by

$\dfrac{x^2}{p_1^2} + \dfrac{y^2}{p_2^2} = \cos^2\theta + \sin^2\theta = 1$, which is an ellipse having p_1 and p_2 for its semi-major and semi-minor axes. This is known as the Ellipse of Stress.

5. Two equal and like Principal Stresses: The case of the two principal stresses being like and equal is interesting. If $p_1 = p_2$, repeating the graphical construction stated above, the two concentric circles will be congruent since $OB = OA$ and proceeding with the construction, the points *E*, *F* and *P* all fall on *F*. *OP* the resultant intensity of stress on the plane *CD* is thus wholly normal to the plane and is equal to *OF* or p_1. The locus of *P*, which was an ellipse when p_1 and p_2 were unequal, now becomes a "Circle of Stress". In this case, every plane is a principal plane and every stress a principal stress of the same intensity as p_1. This is evident if we use the formula $p_n = p_1\cos^2\theta + p_2\sin^2\theta$ and $p_t = (p_1 - p_2)$ $\sin\theta . \cos\theta$.

Since $p_2 = p_1$, p_t is zero and $p_n = p_1(\cos^2\theta + \sin^2\theta) = p_1$.

6. Two equal and unlike Principal Stresses: Repeat the graphical construction and obtain *P'* the mirror image of *P*. *OP'* represents the intensity of stress on the plane *CD*. It is numerically equal to *OP* or p_1, but inclined at angle 2θ to be normal.

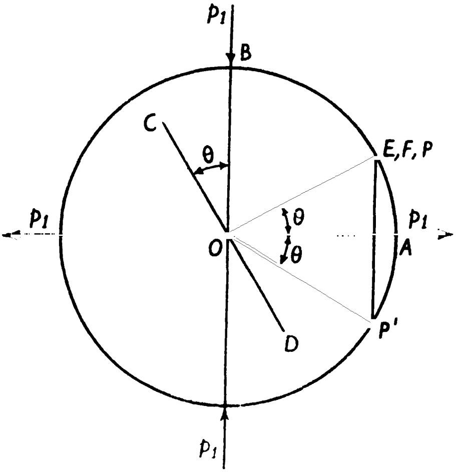


FIG. 18

7. More graphical methods: Another neat graphical method to obtain the resultant intensity of stresses p on the plane *CD* due to two unequal but like principal stresses p_1 and p_2, now suggests itself.

We can express p_1 and p_2 as $(\frac{p_1 + p_2}{2}) + (\frac{p_1 - p_2}{2})$ and $(\frac{p_1 + p_2}{2}) - (\frac{p_1 - p_2}{2})$ respectively.

The principal stresses p_1 and p_2 can thus be split up into two pairs, one pair being $(\frac{p_1 + p_2}{2})$ both equal and like and the other pair being $(\frac{p_1 - p_2}{2})$ both equal but unlike. The effect of the first pair on the plane *CD* will be a wholly normal stress equal to $(\frac{p_1 + p_2}{2})$, while, that of the second will be a stress $(\frac{p_1 - p_2}{2})$ inclined at an angle 2θ to the normal. The total effect of the stresses p_1 and p_2 on the plane will be the vector sum of a stress $(\frac{p_1 + p_2}{2})$ along the normal and a stress $(\frac{p_1 - p_2}{2})$ at 2θ with the normal. The construction is therefore evident. Along the normal to the plane *CD* at *O*, set off *OR* to represent $(\frac{p_1 + p_2}{2})$. At an angle 2θ with *OR*, set off *RP* to scale, to represent $(\frac{p_1 - p_2}{2})$. Then *OP* represents the resultant intensity of stress. If *PQ* is drawn perpendicular to *OR*, *OQ* and *QP* represent p_n and p_t respectively.

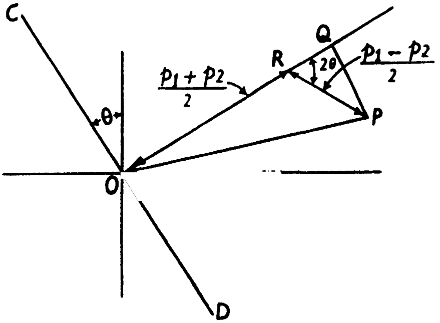


FIG. 19

This construction could have suggested itself also by consideration of the expressions for p_n and p_t.

$$p_n = p_1\cos^2\theta + p_2\sin^2\theta = (\frac{p_1 + p_2}{2}) + (\frac{p_1 - p_2}{2}).\ \cos 2\theta$$

$$p_t = (p_1 - p_2)\ \sin\theta.\cos\theta = (\frac{p_1 - p_2}{2})\ \sin 2\theta.$$

In figure 19,

$$OQ = OR + RQ = \left(\frac{p_1+p_2}{2}\right) + \left(\frac{p_1-p_2}{2}\right)\cos 2\theta, \text{ which is } p_n$$

and $QP = RP\sin 2\theta = \left(\frac{p_1-p_2}{2}\right).\ \sin 2\theta$, which is p_t.

The graphical construction known as "Mohr's Circle" is self-evident. Along the axis OX, set off OA and OB to scale to represent p_1 and p_2 respectively. Bisect BA at C and with C as centre and AB as diameter describe a circle. Set off CP at 2θ with OX to meet the circle at P. Join OP, which represents the resultant intensity of stress on a plane inclined at $\angle\theta$ to the plane of the principal stress p_1.

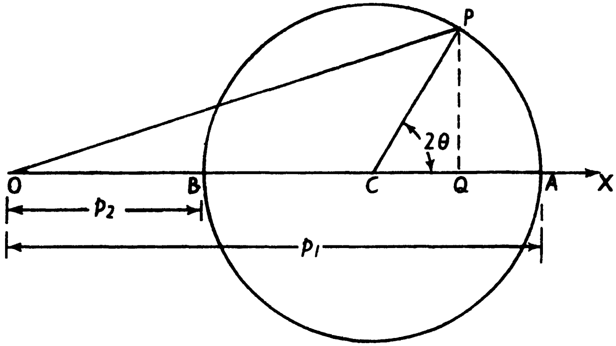


FIG. 20

In figure 20,

$$CA = CB = \frac{p_1-p_2}{2} \text{ and } OC = \frac{p_1+p_2}{2}.$$

From P draw PQ perpendicular to OA.

Then $OQ = OC + CQ = OC + CP\cos 2\theta$

$$= \left(\frac{p_1+p_2}{2}\right) + \left(\frac{p_1-p_2}{2}\right)\cos 2\theta = p_n$$

and $QP = CP\sin 2\theta = \left(\frac{p_1-p_2}{2}\right)\sin 2\theta = p_t$.

OQ and QP respectively represent p_n and p_t and therefore OP represents the resultant intensity of stress p.

Example 2.

If at a point in a strained material, the principal stresses are p_1 *and* p_2*, locate the plane for which the resultant intensity of stress has maximum obliquity.*

Referring to the Mohr's Circle, the locus of P is a circle with C as centre and radius equal to $\frac{p_1-p_2}{2}$ to some scale. For the maximum value of $\angle POC$, OP must be a tangent to the circle as in fig. 21. If ϕ is the maximum angle of obliquity,

$$\sin\phi = \frac{CP}{OC} = \frac{\left(\frac{p_1-p_2}{2}\right)}{\left(\frac{p_1+p_2}{2}\right)} = \frac{(p_1-p_2)}{(p_1+p_2)}$$

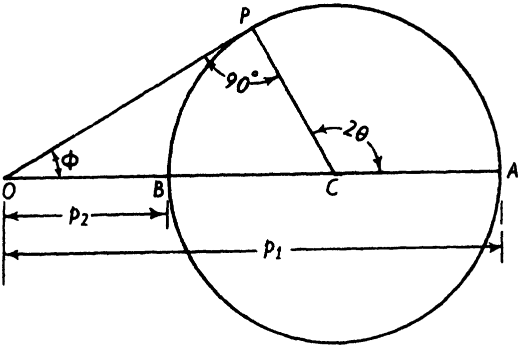


FIG. 21

The position of the plane is given by

$$2\theta = \frac{\pi}{2} + \phi$$

$$\text{i.e.}\quad \theta = \frac{\pi}{4} + \frac{\phi}{2}.$$

This result is useful in the study of Rankine's theory of Earth pressure.

8. Principal stresses and Principal planes: We shall now study a few cases of stresses and strains, locate the principal planes and work out the principal stresses. Ordinarily, every plane sets up normal and tangential resistances and often the values of these stresses are easily calculated. It is, however, not enough merely to know the p_n and p_t for a given plane, but it is necessary to locate the critical planes, viz., those carrying the maximum direct stress and those carrying the maximum shear. In other words, the principal planes and their stresses must be found. The planes of maximum shear and the value of maximum shear are then known.

Case I: At a point in a strained material, the normal intensity of stress p across a plane is accompanied by a state of simple shear q. To find the principal planes and principal stresses.

It is convenient to study a rectangular block *ABCD* of thickness l. Let the intensity of direct stress on faces *AD* and *BC* be p accompanied by a shear stress of intensity q. The faces *AB* and *CD* will only have a shear stress of intensity q.

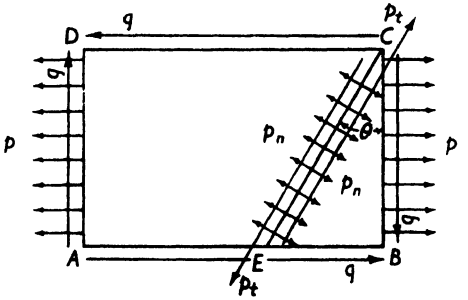


FIG. 22

Consider the normal and tangential intensities of stress set up by any plane *CE* inclined at an angle θ to the plane *BC*. The forces acting on the face *BC* are:

$$P_1 = p.BC.l \rightarrow$$

and $P_2 = q.BC.l \downarrow$

On the face *BE* the force is,

$$P_3 = q.BE.l \rightarrow$$

Resolving the three forces P_1, P_2 and P_3 normally and tangentially to the plane *CE*, we get

$$P_n = (P_1 + P_3)\ \cos\theta + P_2\sin\theta$$

and $P_t = (P_1 + P_3)\ \sin\theta - P_2\cos\theta.$

The normal and tangential intensities of stress across the face *CE* are therefore given by,

$$p_n = \frac{P_n}{CE.l} = \frac{(p.BC.l + q.BE.l)\ \cos\theta + q.BC.l.\sin\theta}{CE.l}$$

$$= p\cos^2\theta + 2q\sin\theta.\cos\theta = \frac{p}{2}\ (1 + \cos 2\theta) + q.\sin 2\theta.$$

And,

$$p_t = \frac{P_t}{CE.l} = \frac{(p.BC.l + q.BE.l)\ \sin\theta - q.BC.l\cos\theta}{CE.l}$$

$$= p.\sin\theta\cos\theta + q.\sin^2\theta - q.\cos^2\theta$$

$$= \tfrac{1}{2}p.\sin 2\theta - q.\cos 2\theta.$$

As expected, the plane *CE* offers both normal and tangential stresses. The principal planes, however, have no tangential stress across them. To locate them, equate p_t to zero.

$$\therefore \qquad \tfrac{1}{2}p.\sin 2\theta - q.\cos 2\theta = 0$$

or

$$\tan 2\theta = \frac{2q}{p}.$$

There are two values of θ which will satisfy this relation, viz., θ_1 and θ_2; the latter will be $(90^\circ + \theta_1)$. There are thus two principal planes which are at right angles to each other, their inclination to the plane of p being given by θ_1 and θ_2 such that

$$\sin 2\theta_1 = \frac{+\,2q}{\sqrt{p^2 + 4q^2}} \text{ and } \cos 2\theta_1 = \frac{+\,p}{\sqrt{p^2 + 4q^2}}$$

and $\sin 2\theta_2 = \dfrac{-\,2q}{\sqrt{p^2 + 4q^2}}$ and $\cos 2\theta_2 = \dfrac{-\,p}{\sqrt{p^2 + 4q^2}}$

To find the values of the principal stresses, substitute for θ, θ_1 and θ_2 in the expression for p_n. If p_1 and p_2 are the principal stresses,

$$p_1 = \frac{p}{2}\ (1 + \cos 2\theta) + q.\sin 2\theta$$

$$= \frac{p}{2} + \frac{p^2}{2\sqrt{p^2 + 4q^2}} + \frac{2q^2}{\sqrt{p^2 + 4q^2}}$$

$$= \frac{p}{2} + \tfrac{1}{2}\sqrt{p^2 + 4q^2}$$

$$= \frac{p}{2} + \sqrt{\left(\frac{p}{2}\right)^2 + q^2}.$$

Similarly,

$$p_2 = \frac{p}{2} - \frac{p^2}{2\sqrt{p^2 + 4q^2}} - \frac{2q^2}{\sqrt{p^2 + 4q^2}}$$

$$= \frac{p}{2} - \frac{1}{2}\sqrt{p^2 + 4q^2} = \frac{p}{2} - \sqrt{\left(\frac{p}{2}\right)^2 + q^2}$$

These are the principal stresses. p_1 is obviously the maximum and is of the same sign as p; p_2 is of the opposite sign, the expression $\sqrt{\frac{p^2}{4} + q^2}$ being greater than $\frac{p}{2}$.

If p_1 is tensile, p_2 must be compressive and vice versa. These planes are shown in figure 23.

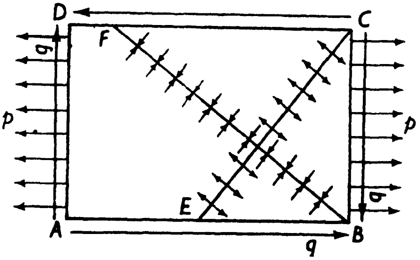


FIG. 23

Once these planes have been located, the planes of maximum shear are easily found. They are mutually at right angles and inclined at 45° to the principal planes. The value of maximum shear is of course $\frac{1}{2}(p_1 - p_2)$ or

$\sqrt{\frac{p^2}{4} + q^2}$, in this case.

The strain in the direction of p_1 will be given by

$$e_1 = \frac{p_1}{E} - \frac{p_2}{mE}$$

i.e. $Ee_1 = p_1 - \frac{p_2}{m}$. The stress which, acting alone, will produce the same maximum strain is thus equal to $p_1 - \frac{p_2}{m}$ and is more than p_1 since p_2 is negative, being of the opposite sign to p_1.

Example 3.

A $\frac{3}{4}''$ diameter bolt is subjected to a pull of 2 tons and a shear force of $\frac{1}{2}$ ton. Calculate the maximum direct and shear stresses induced in the section and specify the position of the planes carrying these stresses with reference to the axis of the bolt. Calculate also the stress which acting alone will produce the same maximum strain. Take $m = 4$.

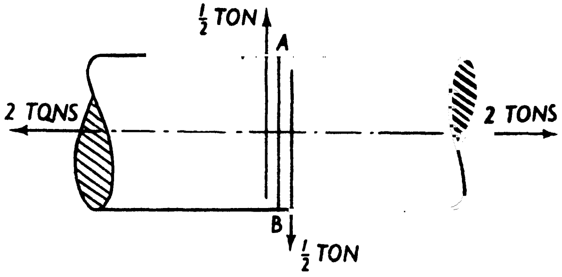


FIG. 24

Consider the normal section AB of the bolt. The direct stress p set up by the section against the pull of 2 tons is given by $p = \frac{2}{\cdot 44} = 4{\cdot}56$ tons/in².

The intensity of shear resistance $q = \frac{\cdot 5}{\cdot 44} = 1{\cdot}14$ tons/in².

If θ is the inclination of a principal plane to AB,

$$\tan 2\theta = \frac{2q}{p} = \frac{2 \times 1{\cdot}14}{4{\cdot}56} = {\cdot}5000$$

$$\therefore \quad 2\theta = 26^\circ 34'$$

$$\therefore \quad \theta_1 = 13^\circ 17'$$

$$\text{and} \quad \theta_2 = 103^\circ 17'.$$

The inclination of these planes to the axis of the bolt will be 76°43′ and 13°17′ respectively. The planes of maximum shear will be inclined at 31°43′ and 58°17′ to the axis.

The principal stresses are given by

$$p_1 = \frac{p}{2} + \sqrt{\frac{p^2}{4} + q^2} = 2{\cdot}28 + \sqrt{(2{\cdot}28)^2 + (1{\cdot}14)^2}$$

$$= 2{\cdot}28 + 2{\cdot}55 = 4{\cdot}83 \text{ tons/in}^2\text{., tensile.}$$

$$p_2 = \frac{p}{2} - \sqrt{\frac{p^2}{4} + q^2} = 2{\cdot}28 - 2{\cdot}55 = -{\cdot}27 \text{ tons/in}^2.$$

or ·27 tons/in². , compressive.

Maximum shear stress $= \dfrac{p_1 - p_2}{2} = 2{\cdot}55$ tons/in².

The stress which acting alone will produce the same maximum strain is given by,

$$p_1 - \frac{p_2}{m} = 4{\cdot}83 + \frac{{\cdot}27}{4} = 4{\cdot}90 \text{ tons/in}^2.$$

Case II: At a point in a strained material, the normal and tangential intensities of stress across two planes at right angles to each other are given. To locate the principal planes and evaluate the principal stresses.

Consider a rectangular block *ABCD* of thickness *l*. Let the intensity of direct stress across the faces *AD* and *BC* be *p* accompanied by a shear intensity *q*. Let *p′* be the intensity of normal stress across the face *AB* and *CD*. The shear intensity across these planes must necessarily be *q*, the same as that across the planes *AD* and *BC*.

To find the normal and tangential stresses offered by a plane *CE* inclined at θ to the face *BC*. The forces acting on the face *BC* are,

$P_1 = p.\ BC.l \rightarrow$

$P_2 = q.\ BC.l \downarrow$

On the face *BE*, they are

$P_3 = q.\ BE.l \rightarrow$

and $P_4 = p'.\ BE.l \downarrow$.

Resolving these forces normally and tangentially to the plane *CE* we get,

$$P_n = (P_1 + P_3) \cos\theta + (P_2 + P_4) \sin\theta$$

$$P_t = (P_1 + P_3) \sin\theta - (P_2 + P_4) \cos\theta.$$

The normal and tangential stress intensities on the plane *CE* are therefore given by,

$$p_n = \frac{P_n}{CE.l} = \frac{(p.BC.l + q.BE.l)\cos\theta + (q.BC.l + p'.BE.l)\sin\theta}{CE.l}$$

$$p\cos^2\theta + q\sin\theta\cos\theta + q\sin\theta\cos\theta + p'\sin^2\theta$$

$$= \frac{p}{2}(1 + \cos2\theta) + \frac{p'}{2}(1 - \cos2\theta) + q.\sin2\theta$$

$$\left(\frac{p + p'}{2}\right) + \left(\frac{p - p'}{2}\right)\cos2\theta + q\sin2\theta.$$

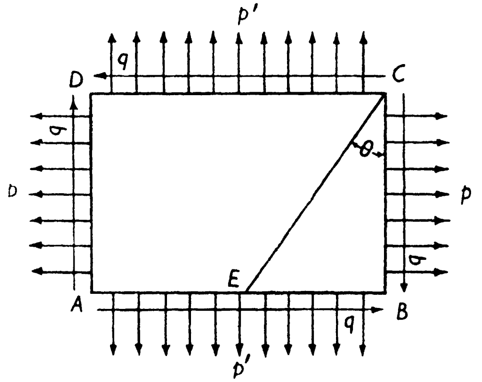


Fig. 25

And,

$$p_t = \frac{P_t}{CE.l} = \frac{(p.BC.l + q.BE.l)\sin\theta - (q.BC.l + p'.BE.l)\cos\theta}{CE.l}$$

$$= p.\sin\theta.\cos\theta + q.\sin^2\theta - q.\cos^2\theta - p'\sin\theta\cos\theta$$

$$\tfrac{1}{2}(p - p')\sin2\theta - q.\cos2\theta.$$

For a principal plane, p_t must be zero.

Equating p_t to zero, we get,

$$\left(\frac{p - p'}{2}\right)\sin2\theta \quad q.\cos2\theta$$

or $$\tan 2\theta = \frac{2q}{p-p'}.$$

There are two values of θ which will satisfy this relation, viz., θ_1 and θ_2, θ_2 being $90° + \theta_1$. There are therefore two principal planes which will be at right angles to each other, their inclination to the plane of p being given by θ_1 and θ_2 such that,

$$\sin 2\theta_1 = \frac{+2q}{\sqrt{(p-p')^2+4q^2}} \text{ and } \cos 2\theta_1 = \frac{+(p-p')}{\sqrt{(p-p')^2+4q^2}}$$

$$\text{Also, } \sin 2\theta_2 = \frac{-2q}{\sqrt{(p-p')^2+4q^2}} \text{ and } \cos 2\theta_2 = \frac{-(p-p')}{\sqrt{(p-p')^2+4q^2}}$$

To evaluate the principal stresses p_1 and p_2, substitute the values of θ_1 and θ_2 for θ in the expression for p_n.

$$\begin{aligned} p_1 &= \left(\frac{p+p'}{2}\right) + \frac{(p-p')^2}{2\sqrt{(p-p')^2+4q^2}} + \frac{2q^2}{\sqrt{(p-p')^2+4q^2}} \\ &= \left(\frac{p+p'}{2}\right) + \tfrac{1}{2}\sqrt{(p-p')^2+4q^2} \\ &= \left(\frac{p+p'}{2}\right) + \sqrt{\left(\frac{p-p'}{2}\right)^2 + q^2}. \end{aligned}$$

And,

$$\begin{aligned} p_2 &= \left(\frac{p+p'}{2}\right) - \frac{(p-p')^2}{2\sqrt{(p-p')^2+4q^2}} - \frac{2q^2}{\sqrt{(p-p')^2+4q^2}} \\ &= \left(\frac{p+p'}{2}\right) - \tfrac{1}{2}\sqrt{(p-p')^2+4q^2} \\ &= \left(\frac{p+p'}{2}\right) - \sqrt{\left(\frac{p-p'}{2}\right)^2 + q^2}. \end{aligned}$$

These are the principal stresses. They need not be unlike as in case I. They may be like or unlike, depending on the values of p, p' and q. The planes of maximum shear will be inclined at 45° to the principal planes, the value of the maximum shear being $\left(\frac{p_1-p_2}{2}\right)$ or $\sqrt{\left(\frac{p-p'}{2}\right)^2 + q^2}$.

The principal strains e_1 and e_2 are given by,

$$e_1 = \frac{p_1}{E} - \frac{p_2}{mE} \text{ and } e_2 = \frac{p_2}{E} - \frac{p_1}{mE}$$

Example 4.

At a certain point in a strained material, the intensities of normal stress across two planes at right angles to each other are 5 and 2 tons/in^2., both tensile and there is a shear of 2 tons/in^2. across the planes. Locate the principal planes and evaluate the principal stresses. Also calculate the maximum intensity of shear stress and specify its planes.

If θ is the inclination of a principal plane to the plane carrying the normal intensity of stress of 5 tons/in^2.,

$$\tan 2\theta = \frac{2q}{p-p'}, = \frac{2 \times 2}{3} = 1{\cdot}333$$

$$\therefore \quad 2\theta = 53°8'$$

$$\therefore \quad \theta_1 = 26°34' \text{ and } \theta_2 = 116°34'$$

$$p_1 = \left(\frac{p+p'}{2}\right) + \sqrt{\left(\frac{p-p'}{2}\right)^2 + q^2}$$

$$= 3{\cdot}5 + \sqrt{(1{\cdot}5)^2+4} = 3{\cdot}5+2{\cdot}5 = 6 \text{ tons/in}^2\text{., tensile.}$$

$$p_2 = 3{\cdot}5 - 2{\cdot}5 = 1 \text{ ton/in}^2\text{., tensile.}$$

The maximum shear stress is 2·5 tons/in^2., the planes offering it being inclined at 71°34′ and 161°34′ to the plane having the normal stress intensity of 5 tons/in^2.

Example 5.

In a material subjected to strain, the resultant intensity of stress across a certain plane is 6 tons/in^2. (tensile) inclined at 30° to its normal. The normal intensity of stress across the plane at right angles to this one is 4 tons/in^2., tensile. Find the principal stresses and principal planes.

Across the first plane, the normal intensity of stress $p = 6 \cos 30° = 3\sqrt{3} = 5{\cdot}2$ tons/in^2., with a shear intensity $q = 6 \sin 30° = 3$ tons/in^2.

Across the second plane, the normal intensity p' is 4 tons/in^2., and the shear intensity must be 3 tons/in^2.

For the principal planes,

$$\tan 2\theta = \frac{2q}{(p-p')} = \frac{2 \times 3}{5{\cdot}2 - 4} = 5$$

$$\therefore\ 2\theta = 78^\circ 42'$$

$$\therefore\ \theta_1 = 39^\circ 21 \text{ and } \theta_2 = 129^\circ 21'$$

$$p_1 = \left(\frac{p+p'}{2}\right) + \sqrt{\left(\frac{p-p'}{2}\right)^2 + q^2} = 4{\cdot}6 + \sqrt{({\cdot}6)^2 + 9}$$

$$= 4{\cdot}6 + 3{\cdot}06 = 7{\cdot}66 \text{ tons/in}^2\text{., tensile}$$

$$p_2 = 4{\cdot}6 - 3{\cdot}06 = 1{\cdot}54 \text{ tons/in}^2\text{., tensile.}$$

The maximum shear intensity will be 3·06 tons/in^2. across the planes of maximum shear.

9. Principal Strains: In the most general case, there will be three principal stresses p_1, p_2 and p_3 across the three principal planes. The total strain in the direction of p_1 due to all the three stresses will be given by:

$$e_1 = \frac{p_1}{E} - \frac{p_2}{mE} - \frac{p_3}{mE} = \frac{p_1}{E} - \frac{p_2 + p_3}{mE}$$

Similarly, $e_2 = \dfrac{p_2}{E} - \dfrac{p_3 + p_1}{mE}$, in the direction of p_2

and $e_3 = \dfrac{p_3}{E} - \dfrac{p_1 + p_2}{mE}$, in the direction of p_3.

These relations are based on the assumption that p_1, p_2 and p_3 are all like. If some of them are unlike, the same expressions may be used with the sign of the unlike stresses changed.

Example 6.

A rectangular block of steel is subjected to stresses of 7 tons/in^2. tensile, 4 tons/in^2. compressive, and 3 tons/in^2. tensile across the three pairs of faces. If m = 4 and E = 13000 tons/in^2., calculate the strain in each of the three directions.

$$e_1 = \frac{7}{E} + \frac{4}{mE} - \frac{3}{mE} = \frac{1}{E}\left(7 + 1 - \frac{3}{4}\right) = \frac{7{\cdot}25}{13000}$$

$$= {\cdot}000558, \text{ tensile.}$$

$$e_2 = \frac{4}{E} + \frac{7}{mE} + \frac{3}{mE} = \frac{1}{E}\left(4 + \frac{7}{4} + \frac{3}{4}\right)$$

$$= \frac{6\cdot5}{13000} = \cdot0005, \text{ compressive.}$$

$$e_3 = \frac{3}{E} - \frac{7}{mE} + \frac{4}{mE} = \frac{1}{E}\left(3 - \frac{7}{4} + 1\right) = \frac{2\cdot25}{13000}$$

$$= \cdot000173, \text{ tensile.}$$

✓ *Example 7.*

A mild steel bar 2″ square in section and 6″ long is subjected to an axial thrust of 20 tons. Half the lateral strain is prevented by the application of uniform external pressure of suitable intensity. If $E = 13000$ *tons/in².,* *and* $\frac{1}{m} = \cdot3$, *calculate the alteration in the length of the bar.*

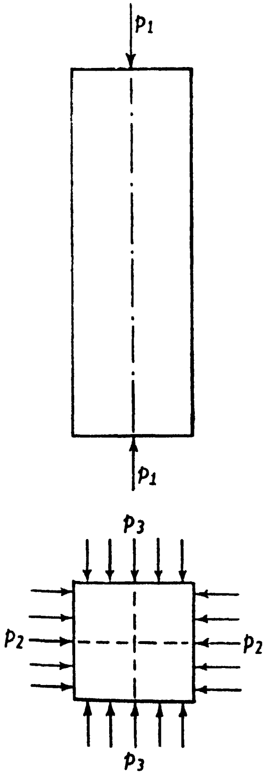


FIG. 26

p_1 the axial stress on the normal cross-section of the bar will be given by $p_1 = \frac{20}{4} = 5$ tons/in².

Let p_2 and p_3 be the stresses along the sides. By symmetry $p_2 = p_3$.

The strain along the axis of the bar is

$$e_1 = \frac{p_1}{E} - \frac{p_2 + p_3}{mE} = \frac{1}{E}(5 - \cdot3 \times 2p_2)$$

$$= \frac{1}{E}(5 - \cdot6p_2)$$

The stretch of either side is

$$e_2 = \frac{p_1}{mE} + \frac{p_3}{mE} - \frac{p_2}{E}.$$

This is one-half of what it would have been if there had been no side pressure p_2 and p_3.

$$\therefore \quad \frac{p_3 + p_1}{mE} - \frac{p_2}{E} = \tfrac{1}{2} \times \frac{p_1}{mE}$$

$$\therefore \quad (p_3 + p_1) - mp_2 = \tfrac{1}{2}p_1$$

$$\text{or} \quad p_2(m - 1) = \tfrac{1}{2}p_1$$

i.e. $p_2 = \dfrac{p_1}{2(m-1)} = \dfrac{5 \times 3}{2 \times 7} = \dfrac{15}{14}$ tons/in².

and $e_1 = \dfrac{1}{E}(5 - \cdot 6p_2) = \dfrac{1}{E}(5 - \dfrac{9}{14})$

$= \dfrac{61}{14E} = \dfrac{61}{14 \times 13000}$

$= \cdot 000335.$

The contraction of length $\delta l = e_1 l_1 = \cdot 000335 \times 6$

$= \cdot 00201''.$

Example 8.

A metal specimen is subjected to an axial compressive stress of p_1. No constraint is exerted on one pair of sides but on the other, external pressures restrict the lateral strain to one-third of that on the first pair. Work out an expression for the axial strain in terms of p_1 and Poisson's ratio $\dfrac{1}{m}$.

Let the external pressures on one pair of sides be p_2. On the other pair, the pressure p_3 is zero because no constraint is exerted.

The axial strain $e_1 = \dfrac{p_1}{E} - \dfrac{p_2}{mE}$.

- The stretch in the direction of p_2 is $e_2 = \dfrac{p_1}{mE} - \dfrac{p_2}{E}$.

The stretch in the direction of p_3 is $e_3 = \dfrac{p_1}{mE} + \dfrac{p_2}{mE}$.

e_2 has been given as $\frac{1}{3}$ of e_3

$\therefore \quad \dfrac{p_1}{mE} - \dfrac{p_2}{E} = \frac{1}{3} \left(\dfrac{p_1 + p_2}{mE}\right)$

or $\quad 3p_1 - 3mp_2 = p_1 + p_2$

$\therefore \quad p_2(3m + 1) = 2p_1$

$\therefore \quad p_2 = \dfrac{2p_1}{(3m + 1)}$

The axial strain

$$e_1 = \frac{p_1}{E} - \frac{p_2}{mE} = \frac{p_1}{E}\left\{1 - \frac{2}{m\,(3m + 1)}\right\}$$

$$= \frac{p_1}{E}\left\{\frac{3m^2 + m - 2}{m\,(3m + 1)}\right\} = \frac{(m + 1)\;(3m - 2)}{m\,(3m + 1)} \times \frac{p_1}{E}.$$

EXAMPLES II

(1) In a boiler plate, the principal stresses across two perpendicular planes are 6 and 3 tons/in^2., both tensile. Find the intensity of stress, which, acting alone, would produce the same maximum strain. $\frac{1}{m} = \cdot 3.$ (5·1 tons/in^2.)

(2) At a point in a strained material, the principal stresses are 7 and $3\frac{1}{2}$ tons/in^2., both tensile. Find the resultant stress intensity on a plane inclined at 60° to the axis of the major principal stress.
(6·31 tons/in^2. at 16°6′ with the axis of 7-ton stress.)

(3) The principal stresses at a point in the section of a member are 8 and 3 tons/in^2., both tensile. Locate the plane across which the resultant stress has maximum obliquity.
($\phi = 27°$; $\theta = 58°30'$ with the plane of the 8-ton stress.)

(4) A bar of steel is under a tensile stress of 8000 lb/in^2., at the same time, under a shear stress of 3000 lb/in^2. Find the principal planes and principal stresses. If $m = 4$, find the stress which, acting alone, would produce the same maximum strain.

{Principal planes inclined at 18°26′ and 108°26′ to the plane of 8000-lb. stress; $p_1 = 9000$ lb/in^2., tensile. $p_2 = 1000$ lb/in^2., compressive; 9250 lb/in^2.}

(5) Determine the maximum and minimum intensities of direct stress and maximum shear stress at a point in the cross-section of a beam with a normal intensity of stress of 5 tons/in^2., compressive, and a shear intensity of $3\frac{1}{2}$ tons/in^2.

{$p_1 = 6{\cdot}8$ tons/in^2., compressive; $p_2 = 1{\cdot}8$ tons/in^2., tensile. max. shear = 4·3 tons/in^2.; $\theta_1 = 27°14'$, $\theta_2 = 117°14'$.}

(6) A material is subjected to a shear stress $q = 3$ tons/in^2., and a tensile stress $p = 4$ tons/in^2. across a plane through a point and a compressive stress $p' = 2\frac{1}{2}$ tons/in^2. across a plane through the point at right angles to the first one. What is the shear stress on the second plane? Determine the magnitudes of the greatest and least intensities of direct stress in the material and the position of their planes; also the position of the planes carrying the maximum shear and its value.

{$q = 3$ tons/in^2.; $\theta_1 = 21°21'$; $\theta_2 = 111°21'$; $p_1 = 5{\cdot}17$ tons/in^2., tensile; $p_2 = 3{\cdot}67$ tons/in^2. compr; max. shear = 4·42 tons/in^2.}

(7) At a certain point in a strained material, the resultant intensity of stress across a plane is 5·66 tons/in^2., (tensile) inclined at

45° to its normal. The normal component of the intensity of stress across the plane at right angles is 2·5 tons/in²., tensile. Find the position of principal planes and stresses across them. Find also the value of the maximum shear at the point.

$$\left\{\begin{array}{l}\theta_1 = 39^\circ 42;\ \ \theta_2 = 129^\circ 42';\ \ p_1 = 7{\cdot}32\ \text{tons/in}^2.,\ \text{tensile;} \\ p_2 = 0{\cdot}82\ \text{tons/in}^2.,\ \text{compressive; max. } p_t = 4{\cdot}07\ \text{tons/in}^2.\end{array}\right\}$$

(8) A bar of elastic material is subjected to a direct compressive stress of p_1 in a longitudinal direction. The lateral strain in the two directions at right angles is entirely prevented by the application of external pressures of suitable intensity. Evaluate the pressure and calculate the ratio of p_1 and the strain e_1 in its own direction.

$$\left\{p_2 = \frac{p_1}{(m-1)}\ ;\ \frac{p_1}{e_1} = \frac{m^2-m}{m^2-m-2} \times E.\right\}$$

(9) A circle 1″ in diameter is marked on a steel plate before it is stressed as shown in fig. 27. As a result of the stresses, the circle deforms to an ellipse. Calculate the lengths of the major and minor axes of the ellipse and their directions Take $E = 13000$ tons/in². and $m = 4$.

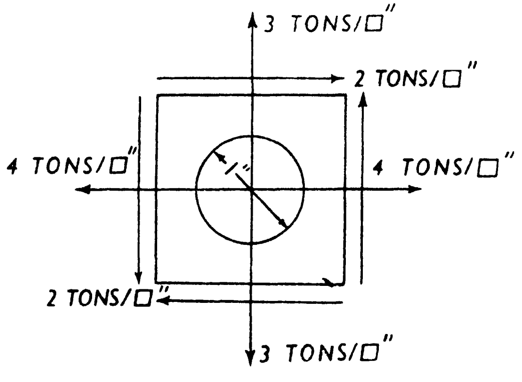


Fig. 27

{ 38° and 52° to the 4-ton stress
Major axis = 1·0004″
Minor axis = 1·000004″. }

(10) A rectangular block of material is subjected to a direct compressive stress of p_1 in a longitudinal direction. The lateral strains in the two directions at right angles are reduced to one-half and one-fourth respectively of those which normally would have occured, by the application of external pressures on the sides of intensity p_2 and p_3 respectively. Evaluate p_2 and p_3 in terms of p_1 and calculate the ratio of p_1 and the strain e_1 in its own direction.

$$\left\{p_2 = \frac{(2m+3)}{4\,(m^2-1)} \times p_1;\ p_3 = \frac{(3m+2)}{4\,(m^2-1)} \times p_1;\ \frac{p_1}{e_1} = \frac{4m\ (m-1)}{4m^2-4m-5} \times E.\right\}$$

(11) At a point in a piece of elastic material, there are three mutually perpendicular planes on which the stresses are as follows: A normal tensile stress of 5 tons/in². and a shearing stress of 4 tons/in². on one plane, a normal compressive stress of 3½ tons/in². and the complementary shearing stress of 4 tons/in². on a second plane and no stress on the third plane. Determine (a) the principal stresses at the point and the position of the planes on which they act and (b) the positions of the planes on which there is no normal stress.

Make diagrams to illustrate your answers, showing the planes and stresses which have been found and their positions relative to the given planes and stresses. (L.U.)

$$\left\{\begin{array}{ll}\text{(a)} & p_1 = 6\cdot59 \text{ tons/in}^2\text{., tensile; } p_2 = 5\cdot09 \text{ tons/in}^2\text{., compr.} \\ & 21^\circ\ 38' \text{ and } 111^\circ\ 38' \text{ with the first plane.} \\ \text{(b)} & 66^\circ\ 38' \text{ and } 156^\circ\ 38' \text{ with the first plane.}\end{array}\right\}$$

(12) An axial compressive load of 50 tons is applied to a metal bar of square section 2″ × 2″. The contraction on an 8″ gauge length is found to be 0·022″ and the increase in thickness 0·00018″. Find the values of Young's modulus and Poisson's ratio.

If in addition to the axial load of 50 tons, a uniform lateral pressure of 5 tons/in². is applied to the two opposite sides of the bar in both directions, find the contraction on the 8″ gauge length and the change in thickness. (L.U.)

$$\left(E = 4545 \text{ tons/in}^2\text{.; } \frac{1}{m} = \cdot327\text{; } \delta l = \cdot01625''\text{; } \delta b \quad \cdot0003212''\text{.}\right)$$

(13) A right angled triangle *ABC* with the right angle at *C* represents planes in an elastic material. There are shearing stresses of 3 tons/in². acting along the planes *AC* and *CB* towards *C* and normal tensile stresses on *AC* and *CB* of 5 tons/in². and 4 tons/in². respectively. There is no stress on the plane perpendicular to the planes *AC* and *CB*.

Determine the positions of the plane *AB* when the resultant stress on *AB* has (a) the greatest magnitude (b) the least magnitude (c) the greatest component normal to *AB* (d) the greatest tangential component along *AB* (e) the least inclination to *AB*.

State for each plane found its angular position relative to *AC* and the magnitude of the stress referred to. (L.U.)

$$\left\{\begin{array}{l}\text{(a) } p_1 = 7\cdot54 \text{ tons/in}^2\text{.; } \theta_1 = 40^\circ\ 18'. \\ \text{(b) } p_2 = 1\cdot46 \text{ tons/in}^2\text{.; } \theta_2 = 130^\circ\ 18'\text{; (c) same as (a).} \\ \text{(d) } 3\cdot04 \text{ tons/in}^2\text{.; } \theta = 85^\circ\ 18' \text{ or } 175^\circ\ 18'\text{; (e) } 106^\circ\ 33'.\end{array}\right\}$$

(14) Describe the Mohr circular diagram of stress as applied to two-dimensional stress, indicating clearly what the principal dimensions and angles on the diagram represent.

At a point in a material subjected to two-dimensional stress, the stresses on a certain plane are 7·5 tons/in^2., tension, and 5 tons/in^2. shear, and on another plane the stresses are 4·5 tons/in^2. tension and 4 tons/in^2. shear as shown in fig. 27(*a*). Find the principal stresses and their directions relative to the given planes. (L.U.)

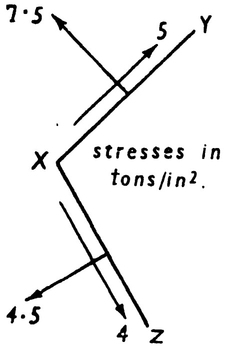


FIG. 27(*a*)

{ $p_1 = 12·5$ tons/in^2. tension; $p_2 = 2·5$ tons/in^2. tension; planes *XY*, *XZ* inclined at 45° and 116° 34′ to the major principal plane. }

(15) In the sketch shown in fig. 27(*b*) *PQ* and *RS* are two planes inclined to one another at 60°. On the plane *PQ* there is a compressive stress of 4 tons/in^2. and a shearing stress of 2 tons/in^2. while on *RS* there is a tensile stress of 1 ton/in^2. and a shearing stress. Determine the value of this shearing stress, the principal stresses and the positions of the principal planes. (L.U.)

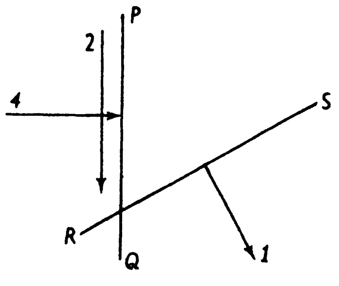


FIG. 27(*b*)

{ $q = 4·89$ tons/in^2.; $p_1 = 4·43$ tons/in^2., compressive, $p_2 = 5·405$ tons/in^2., tensile; *PQ* inclined at 12° to principal plane. }

CHAPTER III

IMPACT OR SHOCK LOADING: STRAIN-ENERGY

1. **Introductory:** Within the elastic limit, stress is proportional to strain and we are now fairly familiar with the use of this relation. In the problems worked out so far, the loading was supposed to be gradual. It may, however, be sudden or may come on a structural member with impact and, in such cases, it is necessary to estimate the maximum instantaneous deformation and the maximum instantaneous stress or resistance accompanying it. This is best done by considerations of the principle of work. When a member is loaded, whether gradually, suddenly, or by impact or shock, work is done upon it. If the elastic limit is not exceeded, this work is stored in the member as strain-energy. It will be given up as soon as the loading is removed, the material behaving like a perfect spring. It is interesting to work out an expression for the strain-energy or energy stored in a member in terms of the maximum stress induced and the dimensions of the member. The expression is very useful because by equating it to the work supplied by the load, whether gradual, sudden or impact, we obtain the value of the maximum stress set up. Since the elastic limit is not exceeded, the value of the maximum instantaneous strain, and so of the deformation for a given length is easily deduced.

2. **Strain-Energy. Resistance-deformation diagram:** Consider a member of length l and uniform sectional area A. When it is loaded, deformation takes place and there is resistance or stress accompanying it. Let δl be the total deformation and p the corresponding intensity of stress, the final resistance R being $p.A$. The final strain $e = \frac{\delta l}{l} = \frac{p}{E}$, since the elastic limit is not exceeded.

Initially when the loading commences, the deformation is zero. Therefore the strain is zero and so the stress or resistance is zero. When the deformation is x, the strain $e_x = \frac{x}{l}$, the corresponding intensity of stress being, $p_x = E.e_x = \frac{E.x}{l}$ and the resistance of the section is $R_x = E.A.\frac{x}{l}$.

It will thus be seen that as x increases R_x increases by a straight line law until when $x = \delta l$, R_x is $R = p.A$.

Plotting R_x against the deformation x, we get the resistance-deformation diagram, which is a straight line.

The amount of work absorbed by the resistance during the deformation is obviously the area under the resistance-deformation diagram and is represented by the triangle OBC in fig. 28.

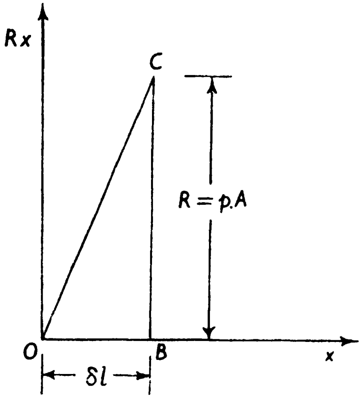


FIG. 28

Energy stored, is therefore,

$$= \tfrac{1}{2}OB.BC$$
$$= \tfrac{1}{2}R.\delta l$$
$$= \tfrac{1}{2}p.A.\delta l$$
$$= \tfrac{1}{2}p.A.el$$
$$= \tfrac{1}{2}p.e.A.l$$
$$= \tfrac{1}{2}\text{ stress} \times \text{strain} \times \text{volume}$$

of the member.

If it is desired to have the expression in terms of the maximum stress induced and the dimensions of the bar, for e substitute $\frac{p}{E}$ and the expression for the energy stored becomes $\frac{1}{2} \times p \times \frac{p}{E} \times A.l$ or $\frac{p^2}{2E} \times$ volume of the loaded member.

If the deformation δl is an extension, the stress p is tensile; if it is a contraction, p is compressive.

3. Gradual, sudden, impact, shock loading:

Case I: Axial load P gradually applied to a bar of length and uniform sectional area A.

Since the load is gradually applied, the work done on the bar = average load × deformation

$$= \tfrac{1}{2} P.\delta l.$$

Work stored during the deformation

$$= \tfrac{1}{2} R.\delta l = \tfrac{1}{2} p.A.\delta l.$$

Equating the work supplied to the work stored, we get

$$\tfrac{1}{2} P.\,\delta l = \tfrac{1}{2} p.A.\delta l$$

or $P = p.A$

and $p = \dfrac{P}{A}$

Case II: Load P suddenly applied.

As the value of the load is P throughout the deformation, the work done on the bar $= P.\delta l$.

The energy stored during the deformation

$$\begin{aligned} &= \tfrac{1}{2}R.\delta l \\ &= \tfrac{1}{2}p.A.\delta l. \end{aligned}$$

Equating,
$$\begin{aligned} P.\delta l &= \tfrac{1}{2}R.\delta l \\ &= \tfrac{1}{2}p.A.\delta l \end{aligned}$$

or $P = \tfrac{1}{2}p.A$

i.e. $p = \dfrac{2P}{A}$

The intensity of stress in this case is twice what it would have been if the load P had been applied gradually. The maximum instantaneous strain is likewise double that in the case of gradual loading.

Case III: Load P dropped through a height h, before it commences to stretch the bar.

If δl is the resulting extension, the work supplied = $P\,(h + \delta l)$, neglecting the loss of energy at impact.

Equating this to the work stored, which is $\tfrac{1}{2}R.\delta l$, we get $P(h + \delta l) = \tfrac{1}{2}R.\delta l = \tfrac{1}{2}p.A.\delta l.$

For δl, substitute $e.l$ i.e. $\frac{p.l}{E}$

$$\therefore \quad P\left(h+\frac{p.l}{E}\right)=\frac{p^2}{2E}\times A.l.$$

This is a quadratic in p which can be re-arranged as

$$p^2=\frac{2Pp}{A}+\frac{2E.P.h}{A.l}$$

or
$$\left(p-\frac{P}{A}\right)^2=\frac{P^2}{A^2}+\frac{2E.P.h}{A.l}$$

$$\therefore \quad p=\frac{P}{A}+\sqrt{\frac{P^2}{A^2}+\frac{2E.P.h}{A.l}}$$

If $\frac{P^2}{A^2}$ is small as compared to $\frac{2E.P.h}{A.l}$ the expression for the maximum instantaneous stress p reduces to

$$p=\frac{P}{A}+\sqrt{\frac{2E.P.h}{A.l}} \text{ or simply to } \sqrt{\frac{2E.P.h}{A.l}}$$

Once the intensity of stress p has been found, the corresponding instantaneous deformation δl is easily deduced, because the strain $e=\frac{p}{E}$ and $\delta l=e.l=\frac{p.l}{E}$.

Case IV: Shock loading.

If U units of energy are transmitted by shock, they will be absorbed by the bar and stored provided the elastic limit is not exceeded. In this case the work supplied by shock $= U$. Equating it to the energy stored by the bar

$$U=\tfrac{1}{2}R.\delta l=\tfrac{1}{2}p.A.\frac{p.l}{E}=\frac{p^2}{2E}\times A.l.$$

Solve for p, which is the maximum instantaneous stress induced in the bar by the shock. The corresponding deformation $\delta l=\frac{p.l}{E}$.

We have seen that the maximum intensity of stress and the corresponding strain with a suddenly applied load are double those due to gradual loading. With impact loading they are still greater.

The relative deformations in the three cases of loading are shown in fig. 29. In the case of suddenly applied loads and impact loading, the bar after undergoing the maximum instantaneous elongation vibrates about *XX*—the position corresponding to a gradually applied load—to which it finally settles down.

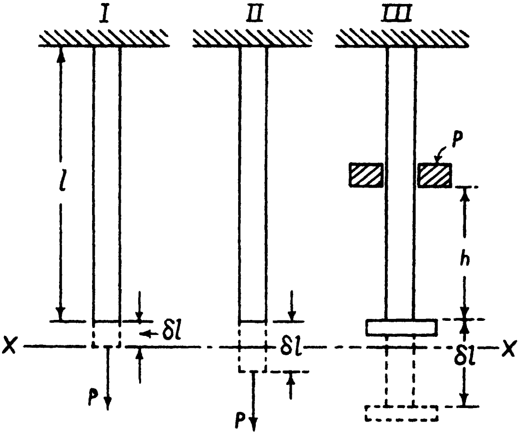


FIG. 29

4. Proof Resilience: The elastic strain energy stored by a member is also known as "Resilience". The greatest strain energy which can be stored by a member without being permanently strained is known as its "Proof Resilience." If f is the stress at the elastic limit for a material i.e. its "Proof Stress", the proof resilience will be $\frac{f^2}{2E} \times$ volume. $\frac{f^2}{2E}$ or the proof resilience per unit volume can be looked upon as a property of the material and may be called its "Modulus of Resilience".

Example 1.

A steel rod 1½" in diameter is 10 feet long. Find the work done when an axial pull of 8 tons is suddenly applied to it. Calculate the maximum instantaneous stress and elongation produced. $E = 13500$ *tons/in*^{2}.

Area of section $A = 1{\cdot}77$ in^2. If δl is the maximum instantaneous elongation, work done $= P.\delta l$.

$$\text{Work stored} = \tfrac{1}{2}R.\delta l$$
$$= \tfrac{1}{2}p.A.\delta l.$$

$$\text{Equating,} \quad p = \frac{2P}{A} = \frac{2 \times 8}{1{\cdot}77} = 9{\cdot}04 \text{ tons/in}^2.$$

$$\text{The corresponding strain } e = \frac{p}{E} = \frac{9{\cdot}04}{13500} = {\cdot}000669$$

$$\delta l = e.l = {\cdot}000669 \times 120$$
$$= {\cdot}08''.$$

The work done on the rod and stored by it

$$= P.\delta l = 8 \times {\cdot}08 = {\cdot}64 \text{ inch-ton.}$$

Example 2.

A weight of 50 lb. is dropped onto a collar at the end of a vertical bar 6 feet long and 1″ in diameter, from a height of 4″. Calculate the maximum instantaneous extension and stress produced in the section of the bar. E = 30 × 10⁶ lb/in².

Work supplied = $P(h + \delta l)$, neglecting the loss of energy at impact.

$$\text{Work stored} = \frac{p^2}{2E} \times A \times l.$$

$$\text{Equating,} \quad P\left(h + \frac{p.l}{E}\right) = \frac{p^2}{2E}.A.l$$

$$\text{or} \quad p^2 - \frac{2P}{A}.p = \frac{2E.P.h}{A.l}$$

$$\therefore \quad p^2 - 127{\cdot}3p = \frac{2 \times 30 \times 10^6 \times 63{\cdot}65 \times 4}{72}$$

$$\therefore \quad = 212{\cdot}1 \times 10^6$$

$$(p - 63{\cdot}65) = 14{\cdot}56 \times 10^3$$

$$\therefore \quad p = 14620 \text{ lb/in}^2.$$

$$e = \frac{p}{E} = \frac{14620}{30 \times 10^6} = {\cdot}000487$$

$$\delta l = e.l = {\cdot}000487 \times 72 = {\cdot}035''.$$

Example 3.

A weight of ~~2 cwt.~~ is dropped onto a collar at the lower end of a vertical bar 10 feet long and 1¼″ in diameter. Calculate the height *of drop if the maximum instantaneous stress is not to exceed 8 tons/in^2. What is the corresponding instantaneous elongation? E = 13000 tons/in^2.*

Max. instantaneous strain $e = \frac{p}{E} = \frac{8}{13000} = \cdot 000615.$

Max. instantaneous elongation

$$\delta l = e.l = \cdot 000615 \times 120$$
$$= \cdot 074''.$$

Work supplied by the falling weight

$$= P\ (h + \delta l) = \frac{1}{10}\,(h + \cdot 074) \text{ inch-ton.}$$

Work stored by the bar at 8 tons/in^2.

$$= \frac{p^2}{2E} A.l = \frac{64 \times 1\cdot 23 \times 120}{2 \times 13000} = \cdot 3633 \text{ inch-ton.}$$

Equating the work supplied to work stored,

$$\frac{1}{10}\,(h + \cdot 074) = \cdot 3633$$
$$\therefore \quad h + \cdot 074 = 3\cdot 633$$
$$\therefore \quad h = 3\cdot 56''.$$

Example 4.

A waggon weighing 4 tons is attached to a wire-rope and is moving down an incline at a speed of ~~2 miles~~ per hour when the rope jams and the waggon is suddenly brought to rest. If the length of the rope is ~~150 feet~~ at the time of the sudden stoppage, calculate the maximum instantaneous stress and the maximum instantaneous elongation produced. Diameter of rope = 1½″. E = 13000 tons/in^2. g = 32 ft./sec^2.

The kinetic energy of waggon $= \frac{mv^2}{2g}$

$$= \frac{4 \times 44 \times 44}{15 \times 15 \times 64} = \frac{121}{225} \text{ ft-ton} = 6\cdot 45 \text{ inch-tons.}$$

This is to be absorbed by the rope at a stress p.

$$\text{Energy stored} = \frac{p^2}{2E} \times A.l$$
$$= \frac{p^2}{2E} \times 1\cdot 77 \times 1800$$

$$= \cdot 1226\ p^2 \text{ inch-tons.}$$

Equating, $\cdot 1226\ p^2 = 6\cdot 45$

$$p^2 = 52\cdot 64$$

$$\therefore \quad p = 7\cdot 25 \text{ ton/in}^2.$$

The maximum instantaneous elongation of the rope

$$\delta l = e.l = \frac{p}{E} \times l$$

$$= \frac{7\cdot 25 \times 1800}{13000} = 1 \text{ inch.}$$

Example 5.

A wrought iron bar 2″ in diameter and 8 feet long has to transmit a shock energy of 50 ft-lbs. Calculate the maximum instantaneous stress and elongation produced. $E = 28 \times 10^6$ *lb/in*^{2}.

Work to be stored $= 50$ ft-lbs.

$$= 600 \text{ inch-lbs.}$$

$$\text{Energy stored} = \frac{p^2}{2E} \times \text{volume} = \frac{p^2 \times 3\cdot 1416 \times 96}{2 \times 28 \times 10^6}$$

Equating, $p^2 = 111\cdot 4 \times 10^6$

$$\therefore \quad p = 10550 \text{ lb/in}^2.$$

$$e = \frac{p}{E} = \frac{10550}{28 \times 10^6} = \cdot 000377$$

$$\therefore \quad \delta l = e.l = \cdot 000377 \times 96 = \cdot 0362''.$$

Example 6.

Two bars A and B, of circular section and of the same material, are each 12″ long. A is 1″ in diameter for 4″ of its length and 2″ in diameter for the remaining 8″. B is 1″ in diameter for 8″ of its length and 2″ in diameter for the rest. An axial blow is given to A, which produces a maximum instantaneous stress in it of 7½ tons/in^2. *Calculate the max. instantaneous stress produced by the same blow on B. If each bar is stressed upto the elastic limit, calculate the ratio of energy stored by B and A at proof-stress. Calculate also the ratio per unit volume of each bar.*

Let U_A and U_B be the energies stored by the bars A and B at stresses p_A and p_B respectively.

When the stress is p in the 1″ portion, the corresponding stress in the 2″ portion will obviously be $\frac{p}{4}$, as the stresses are inversely proportional to the areas of resisting sections.

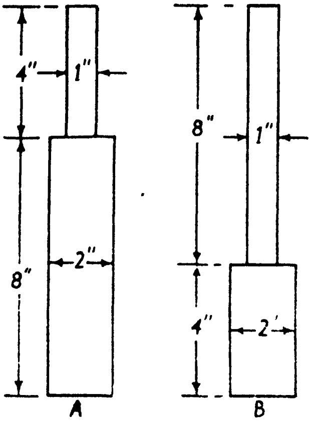


FIG. 30

$$U_A = \frac{p_A^2}{2E} \times \frac{\pi}{4} \times 1^2 \times 4 + \frac{(\frac{1}{4}p_A)^2}{2E} \times \frac{\pi}{4} \times 2^2 \times 8$$

$$= \frac{\pi p_A^2}{2E}(1 + \tfrac{1}{2}) = \frac{3\pi p_A^2}{4E}.$$

$$U_B = \frac{p_B^2}{2E} \times \frac{\pi}{4} \times 1^2 \times 8 + \frac{(\frac{1}{4}p_B)^2}{2E} \times \frac{\pi}{4} \times 2^2 \times 4$$

$$= \frac{\pi p_B^2}{2E}(2 + \tfrac{1}{4}) = \frac{9\pi p_B^2}{8E}$$

If an axial blow is delivered which produces in A a stress p_A of 7½ tons/in²., the same blow will produce in B a stress p_B given by $\frac{9\pi p_B^2}{8E} = \frac{3\pi b_A^2}{4E}$

or $$p_B^2 = \tfrac{2}{3}p_A^2 = \tfrac{2}{3} \times \frac{225}{4} = 37{\cdot}5$$

∴ $$p_B = 6{\cdot}12 \text{ tons/in}^2.$$

If both the bars are allowed to reach the proof stress f, $p_A = p_B = f$ and the ratio of the energies stored by B and A is

$$\frac{U_B}{U_A} = \frac{9\pi}{8E} \times \frac{4E}{3\pi} = \frac{3}{2} = 1{\cdot}50.$$

B therefore stores 50% more energy than A at the same stress. Per unit volume, the energy stored by B will be still higher.

If V_B is the volume of the bar B,

$$V_B = \frac{\pi}{4} \times 1^2 \times 8 + \frac{\pi}{4} \times 2^2 \times 4 = 6\pi \text{ cub-inches.}$$

$$\text{Per unit volume, } U_B = \frac{9\pi}{8E}p_B^2 \times \frac{1}{6\pi} = \frac{3}{16} \times \frac{p_B^2}{E}$$

$$\text{Similarly } V_A = \frac{\pi}{4} \times 1^2 \times 4 + \frac{\pi}{4} \times 2^2 \times 8 = 9\pi \text{ cub-inches.}$$

$$\text{Per unit volume, } U_A = \frac{3\pi}{4E}p_A^2 \times \frac{1}{9\pi} = \frac{1}{12} \times \frac{p_A^2}{E}$$

The ratio of the energies stored at the same stress, per unit volume, is

$$\frac{3}{16E} \div \frac{1}{12E} = \frac{9}{4} = 2{\cdot}25.$$

5. Shear Resilience: If q is the uniform intensity of shear stress produced in a material by external forces, the expression for the energy stored within the elastic limit, works out to $\frac{1}{2N}.q^2 \times$ volume, where N is the shear modulus or modulus of Rigidity.

Consider a square block $ABCD$ of length l, the faces BC and AD being stressed to q. Let AD be fixed. The section $ABCD$ will deform to AB_1C_1D, through the angle φ which represents the shear strain. The force P on the face $BC = q.BC.l$. Work done by P, gradually applied,

$$= \tfrac{1}{2}P \times BB_1 \text{ or } CC_1$$

$$= \tfrac{1}{2}P.AB.\varphi = \tfrac{1}{2}q.BC.l.AB.\varphi$$

$$= \tfrac{1}{2}q.\varphi.AB.BC.l$$

$$= \tfrac{1}{2} \text{ stress} \times \text{strain} \times \text{volume of the block, as before.}$$

If we replace the strain φ by $\frac{q}{N}$, the expression becomes $\frac{1}{2} \times \frac{q^2}{N} \times$ volume.

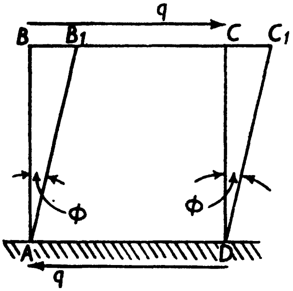


FIG. 31

6. Strain-energy in terms of principal stresses: If at a point in a strained material, the principal stresses are p_1 and p_2, the principal strains are given by

$$e_1 = \frac{p_1}{E} - \frac{p_2}{mE} \text{ and } e_2 = \frac{p_2}{E} - \frac{p_1}{mE}.$$

Per unit volume, the strain-energy will be,

$$\tfrac{1}{2}p_1 . e_1 + \tfrac{1}{2}p_2 . e_2$$
$$= \tfrac{1}{2}p_1\left(\frac{p_1}{E} - \frac{p_2}{mE}\right) + \tfrac{1}{2}p_2\left(\frac{p_2}{E} - \frac{p_1}{mE}\right)$$
$$= \frac{1}{2E}\left(p_1^2 + p^2 - \frac{2p_1p_2}{m}\right).$$

In the general case of three principal stresses p_1, p_2 and p_3, the three principal strains are,

$$e_1 = \frac{p_1}{E} - \frac{p_2 + p_3}{mE}$$
$$e_2 = \frac{p_2}{E} - \frac{p_3 + p_1}{mE}$$
$$e_3 = \frac{p_3}{E} - \frac{p_1 + p_2}{mE}$$

The strain-energy per unit volume is

$$= \tfrac{1}{2}p_1.e_1 + \tfrac{1}{2}p_2.e_2 + \tfrac{1}{2}p_3.e_3$$

$$= \frac{p_1}{2}\left\{\frac{p_1}{E} - \frac{(p_2+p_3)}{mE}\right\} + \frac{p_2}{E}\left\{\frac{p_2}{2} - \frac{(p_3+p_1)}{mE}\right\} + \frac{p_3}{2}\left\{\frac{p_3}{E} - \frac{(p_1+p_2)}{mE}\right\}$$

$$= \frac{1}{2E}\left\{p_1^2 + p_2^2 + p_3^2 - \frac{2p_1p_2 + 2p_2p_3 + 2p_3p_1}{m}\right\}$$

7. Relation between the elastic moduli: The relation between the moduli of Elasticity, Rigidity and Bulk for a given material has already been worked out. It could also be established from considerations of strain-energy.

Consider the case of a state of simple shear of intensity q. The strain-energy per unit volume in this case is $\frac{q^2}{2N}$.

But the state of simple shear is equivalent to two principal stresses, across diagonal planes, of the same intensity q but opposite in sign. Let $p_1 = +q$ and $p_2 = -q$. The expression for strain-energy in terms of the principal stresses p_1 and p_2 is $\frac{1}{2E}(p_1^2 + p_2^2 - \frac{2p_1p_2}{m})$.

Substituting $+q$ for p_1 and $-q$ for p_2, the expression reduces to $\frac{q^2}{E}(1 + \frac{1}{m})$.

Equating,

$$\frac{q^2}{2N} = \frac{q^2}{E}\left(1 + \frac{1}{m}\right)$$

$\therefore \quad E = 2N\left(1 + \frac{1}{m}\right)$ as has already been proved.

When a material is subjected to three direct stresses of the same intensity p across three mutually perpendicular planes, the ratio of p and the resulting volumetric strain has been defined as the Bulk Modulus of the material.

In this case $p_1 = p_2 = p_3 = p$ and the strain-energy per unit volume can be obtained by substituting p for p_1, p_2 and p_3 in the expression,

$\frac{1}{2E}\left\{p_1^2 + p_2^2 + p_3^2 - \frac{2(p_1p_2 + p_2p_3 + p_3p_1)}{m}\right\}$ which reduces to $\frac{3p^2}{2E}(1 - \frac{2}{m})$. If the volume of the material is V, the strain-energy will be $\frac{3p^2}{2E}(1 - \frac{2}{m})\ V$........................(1)

By definition, $K = \frac{p}{\text{vol. strain}}$

$$\therefore \quad \text{vol. strain} = \frac{\delta V}{V} = \frac{p}{K}$$

or
$$\delta V = \frac{p}{K}.V.$$

If a uniform intensity of pressure p is gradually applied to a body of volume V, and δV is the resulting change in volume, the work done

$$= \tfrac{1}{2}p.\delta V$$

$$= \tfrac{1}{2}p.\frac{p}{K}.\ V = \frac{p^2}{2K}.V \text{..............................(2)}$$

Equating (1) and (2),

$$\frac{3p^2}{2E}(1 - \frac{2}{m})\ V = \frac{p^2}{2K}.V$$

or
$$E = 3K\ (1 - \frac{2}{m})$$

The relation $E = 2N\ (1 + \frac{1}{m}) = 3K\ (1 - \frac{2}{m})$ has thus been established.

8. Criterion for design: For a material in a state of stress, we know how to work out the maximum direct stress, the maximum strain, the maximum shear stress, and the strain-energy in terms of the principal stresses. In designing a structural member, care has to be taken that the stress on the proposed section for the member does not exceed the working stress considered suitable for the material. If, for direct stress, f is the working stress and it is f_s for shear, the following criteria for design of the member are available:

(*i*) *Maximum Principal Stress:* The maximum principal stress must not exceed the working stress for the material i.e. $p_1 \leqq f$.

This looks reasonable and was considered a satisfactory criterion for design in the early days of Engineering Science. It is known as the Rankine hypothesis after the pioneer — Professor Rankine.

(*ii*) *Maximum Strain:* If p_1 and p_2 are the principal stresses at a point in the material under stress, the principal strain

$$e_1 = \frac{p_1}{E} - \frac{p_2}{mE}$$

or $Ee_1 = (p_1 - \frac{p_2}{m})$ i.e. the stress which acting alone, will produce the same maximum strain e_1 is given by $(p_1 - \frac{p_2}{m})$.

We have seen that this exceeds the maximum principal stress p_1 if p_1 and p_2 are unlike. The criterion of design will, therefore be, that $(p_1 - \frac{p_2}{m}) \leqq f$.

This theory is associated with St. Venant.

(*iii*) *Maximum Shear Stress:* If p_1 and p_2 are the principal stresses at a point, the maximum shear stress is $\frac{1}{2}\ (p_1 - p_2)$ across mutually perpendicular planes inclined at 45° to the principal planes. The criterion of design can be, $\frac{1}{2}\ (p_1 - p_2) \leqq f_s$, if failure is likely to occur by shear. This theory is associated with the name of Sir J. J. Guest.

(*iv*) *Maximum Strain-energy:* If p_1 and p_2 are the principal stresses, the strain-energy per unit volume is

$$\frac{1}{2E}\ (p_1^2 + p_2^2 - \frac{2p_1p_2}{m}).$$

According to this theory, the maximum strain-energy must not exceed the proof-resilience of the material, i.e.

$$\frac{1}{2E}\left(p_1^2 + p_2^2 - \frac{2p_1p_2}{m}\right) \leqq \frac{f^2}{2E}$$

$$\text{or } \left(p_1^2 + p_2^2 - \frac{2p_1p_2}{m}\right) \leqq f^2.$$

EXAMPLES III

(1) Water under a pressure of 1000 lb/in^2. is suddenly admitted on a plunger of 4½″ diameter, attached to a rod 1¼″ in diameter and 40 feet long. Find the maximum instantaneous stress and deformation of the rod if $E = 30 \times 10^6$ lb/in^2. (25,920 lb/in^2.; ·415″.)

(2) A ½″ diameter mild steel bar of 5 feet length is stressed by a weight of 30 lb. dropping freely through 1½″ before commencing to stretch the bar. Find the maximum instantaneous stress and elongation produced in the bar. $E = 30 \times 10^6$ lb/in^2. (15,300 lb/in^2.; ·03″.)

(3) A 4″×4″×½″ bar of Tee-section is 12′ 6″ long. An axial load of 20 tons is suddenly applied to it. Find (i) the maximum instantaneous stress (ii) the maximum instantaneous elongation and (iii) the work stored in the bar at the instant of maximum elongation. $E = 13000$ tons/in^2. (10·67 tons/in^2.; ·123″; 2·46 in-tons.)

(4) It is found that a bar 1½″ in diameter stretches ·076″ under a gradually applied load of 15 tons. If a weight of 300 lb. is dropped onto a collar at the lower end of this bar, through a height of 2″ before commencing to stretch the bar, calculate the maximum instantaneous stress and elongation produced in the bar. $E = 30 \times 10^6$ lb/in^2. (13,170 lb/in^2.; ·053″.)

(5) A vertical steel rod of uniform diameter, 8′ 4″ long, is subjected to a load of 4 cwt. dropping ¾″ onto a collar at the lower end of the bar. Calculate the diameter of the rod if the maximum strain is not to exceed $\frac{1}{2000}$. $E = 13000$ tons/in^2. (1⅛″ diameter.)

(6) Determine the greatest weight that can be dropped from a height of 8″ onto a collar at the lower end of a vertical bar, 1⅜″ in diameter and 8 feet long without exceeding the elastic limit which is 17 tons/in^2. Calculate also the instantaneous elongation of the bar. $E = 13000$ tons/in^2. (436·4 lb/wt. ·1255″.)

(7) A wrought iron bar 4 feet long, has to transmit a shock of 50 ft-lbs. without injuring its elasticity. What minimum sectional area should be provided if the stress at elastic limit is 14 tons/in^2. and $E = 12500$ tons/in^2.? (·712 sq. in.)

(8) An iron bar, 2″ in diameter, is subjected to a shock of 150 ft-lbs. If the maximum instantaneous stress of 8 tons/in². is not to be exceeded, calculate the minimum length of the bar which must be provided to transmit the shock. $E = 12000$ tons/in². (8 feet.)

(9) A bar of steel 10 feet long, is 1¼″ in diameter for 3 feet of its length, 1⅛″ in diameter for 4 feet and 1″ in diameter for the remaining 3 feet. The bar is in tension, the stress on the smallest section being 7 tons/in². Taking $E = 13500$ tons/in²., find the total elongation of the bar and the energy stored by it.

(·05″: ·138 inch-ton.)

(10) If the bar in question 9, is given an axial blow transmitting a shock energy of 75 ft-lbs., calculate the maximum instantaneous stress induced and the maximum inst. elongation of the bar. $E = 13500$ tons/in². (11·93 tons/in².; ·086″.)

(11) A vertical rod, ¾″ diameter and 10′ long is fixed at one end and is used to stop suddenly a weight P falling in the direction of its length. If P falls 10″ before coming to rest, find its value if the maximum instantaneous stress produced in the bar is 16000 lb/in². $E = 30 \times 10^6$ lb/in². (22·61 lb-wt.)

(12) A crane-chain lowers a load of 2 tons at a uniform rate of 1½ feet per second. When the length of the chain unwound is 25 feet, it suddenly gets jammed. Estimate the instantaneous stress induced in it due to the sudden stoppage and the maximum instantaneous elongation, if each link is O-shaped and made of ½″ diameter steel rod. Estimate also the impulsive pull on each link.
$E = 13000$ ton/in²., $g = 32$ ft./sec².

(18·74 tons/in².; ·432″; 7·36 tons.)

(13) Two round bars A and B are each 12″ long. A is ¾″ in diameter for a length of 4 inches and 1½″ in diameter for the remaining 8″. B is ¾″ in diameter for a length of 8 inches and 1½″ in diameter for the rest. A receives an axial blow which produces a stress of 12 tons/in². Find the max. stress produced by the same blow on B. How much more energy will B absorb than A without exceeding a given stress within the elastic limit?

(9·8 tons/in².; 50% more.)

(14) A vertical round steel rod, 6 feet long, is securely held at its upper end and a weight sliding freely on the rod falls onto a stop at the lower end of the rod. When the weight falls from a height of 1¼″, the maximum stress reached in the rod is estimated to be 10 tons/in².

Determine the stress if the load had been applied gradually and also the maximum stress if the load had fallen from a height of $1\frac{3}{4}''$. $E = 30 \times 10^6$ lb/in². (L.U.)

(461·7 lb/in²; 11·79 tons/in².)

(15) A copper bar $1\frac{1}{2}''$ diameter is enclosed in a steel tube which is $2''$ in external diameter, thickness of metal $\frac{1}{4}''$. The composite bar is held vertically, being rigidly gripped at the top and provided with a collar at the bottom. A weight of 700 lb. is dropped onto the collar from a height of $\frac{3}{4}''$. If the length of the bar is 6 feet, find the maximum instantaneous elongation of the bar and the maximum stress induced in each metal. $E_c=7000$ tons/in².; $E_s=13000$ tons/in².

(·03415″; $p_s=6{\cdot}17$ tons/in².; $p_c=3{\cdot}32$ tons/in².)

CHAPTER IV

BEAMS AND BENDING — I.

Bending Moments and Shear Forces

1. Definitions: A structural member which is acted upon by a system of external forces at right angles to its axis is called a *beam.* Such loading is called transverse loading and the resulting stresses are called transverse stresses.

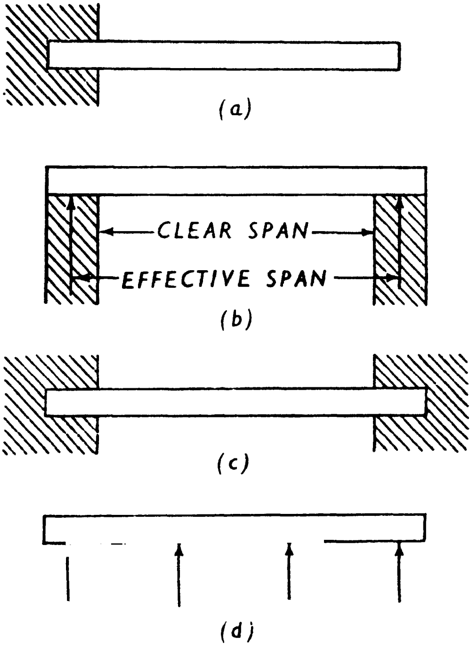


FIG. 32

A beam which is fixed at one end, the other end being free, is called a *cantilever.*

A beam with its ends resting freely on supports which may be walls or columns, is called a *freely supported beam.* The horizontal distance between the supporting walls is called the "*Clear*" span of the beam. The horizontal distance between the lines of action of the end-reactions is called its "*Effective*" span. A beam with its ends built-in into the end walls is called a *fixed*, *built-in* or *encastré beam.*

A beam which has more than two supports is a *continuous* beam, being continuous over intermediate supports. These are illustrated in fig. 32(*a*), (*b*), (*c*) and (*d*).

These distinctions, based on the manner in which a beam is supported, are necessary because it will be seen that the deformations and the bending stresses in a beam depend not merely on its span and the transverse loads on it, but also on how the beam is supported.

2. Actions on the cross-section of a beam: Before we can study the deformations and the resistances set up in a beam, it is first necessary to know the effects of the load-system on a given cross-section of the beam, which the cross-section is called upon to resist.

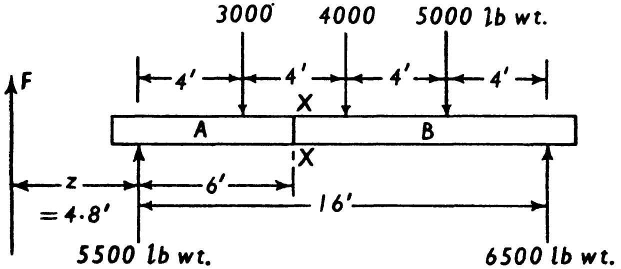


FIG. 33

Consider a beam with an effective span of 16 feet with point-loads of 3000, 4000 and 5000 lb-wt. placed on it at 4, 8 and 12 feet from the left-hand support. We shall investigate the effect of the load system on the cross-section 6 feet from the left-hand support. The reactions are 5500 and 6500 lb-wt. at the left-hand and right-hand supports respectively. Let the cross-section divide the beam into two parts *A* and *B*. Consider the forces acting on *A*.

These are two, viz., the upwards reaction of 5500 lb-wt. at the support and the downward force of 3000 lb-wt. The resultant of these forces is an upward force $F = 2500$ lb-wt.

If the distance of its line of action from the left-hand support is z, by equating the moment of the resultant to the algebraic sum of the moments of the component forces about the left-hand support, we get,

$$\begin{aligned} 2500z &= 5500 \times 0 + 3000 \times 4 \\ &= 12000 \\ \therefore \quad z &= 4{\cdot}8 \text{ feet.} \end{aligned}$$

The effect of the load system on the portion A of the beam is therefore an upward force F of 2500 lb-wt. acting at 4·8 feet from the support as shown in fig. 33. This has to be resisted at the section XX. Introduce two equal and opposite forces F — F at XX. The portion A of the beam has now three forces F on it as shown in fig. 34, which can be grouped into (i) upward force F acting at XX and (ii) a couple formed by the remaining two, whose moment is $F \times (4{\cdot}8 + 6) = 10{\cdot}8F$

$= 10{\cdot}8 \times 2500 = 27000$ lb-ft. clockwise.

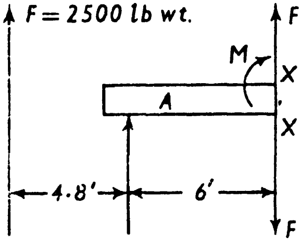


FIG. 34

The effect of the load system on the portion A of the beam, to be resisted at the section XX is seen to be twofold, viz., (i) an upward force F = 2500 lb-wt. acting across the cross-

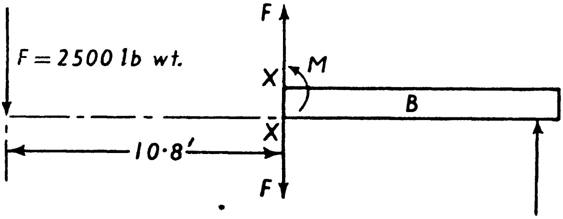


FIG. 35

section and (ii) a couple $M = 27000$ lb-ft. If we study the forces acting on the portion B of the beam, the resultant is a downward force $F = 2500$ lb-wt., its line of action being the same as that of the resultant force on the portion A. This is equivalent to (i) a downward force $F = 2500$ lb-wt. acting at the section XX and (ii) a couple $M = F \times (4{\cdot}8+6) = 10{\cdot}8F =$ 27000 lb-feet anticlockwise as shown in fig. 35.

The effect of the load system at the cross-section XX can now be marked as in fig. 36. The forces $F - F$ are equal and opposite and act across the cross-section, tending to shear it off. The section has therefore to set up shear stresses to resist the action of these forces. The couples $M — M$ acting on the section tend to bend the beam at the cross-section which has to set up bending stresses to resist these couples.

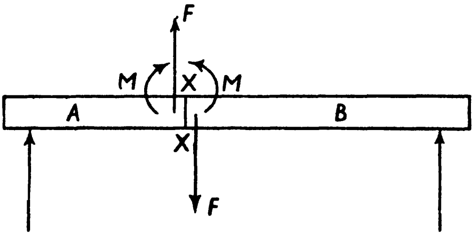


FIG. 36

The force F is called the Shear Force and the couple M is called the Bending Moment. The effect of the load-system on the cross-section of a beam, therefore, consists of (i) the Shear Force and (ii) the Bending Moment.

The value of the shear force is easily obtained. It is the unbalanced vertical force on the portion A or B of the beam. To obtain the Bending Moment, it might seem as if it would be necessary to locate the line of action of the resultant force F on the portion A or B of the beam, to be able to get the arm of the couple constituting the Bending Moment.

Actually, it is not necessary. The moment of a couple about any point is constant. Referring to fig. 33, and taking moments about the cross-section XX itself, the moment of the couple M is equal to the moment of the resultant F about XX. This must be equal to the algebraic sum of the moments of the component forces about XX.

$$\begin{aligned} \text{Thus,}\quad M &= 5500 \times 6 - 3000 \times 2 \\ &= 33000 - 6000 = 27000 \text{ lb-ft. as before.} \end{aligned}$$

Therefore, to obtain the Bending Moment at a cross-section, all we need do, is to take the moments of the forces on *A or B* about the cross-section. Their algebraic sum constitutes the Bending Moment. We may therefore make the following statement:

(i) The shear force at a cross-section of a beam is the unbalanced vertical force to the right *or* left of the section.

(ii) The Bending Moment at a cross-section of a beam is the algebraic sum of the moments of the forces to the right *or* left of the section.

3. S.F. and B.M. diagrams: These actions at the cross-section of a beam, viz., the Shear Force and the Bending Moment will be different at different cross-sections, depending upon the loading on the beam. They can be calculated numerically for any given cross-section or algebraical expressions can be worked out for them, in terms of the loads and the position of the section. The variation in their values can be shown graphically by plotting the S.F. or the B.M. as ordinates against the positions of the cross-sections as abscissae. These are known as S.F. and B.M. diagrams. We shall now construct a few of these diagrams for beams with a given system of loading on them. We must particularly note the sections subjected to maximum S.F. and maximum B.M. because the corresponding stresses which these sections will have to set up to resist the actions, will be maximum.

I. Cantilevers:

(i) Span l, end-load W.

Consider a section X at a distance of x from the free end B. Shear force $F_x = W$. Bending moment $M_x = W.x$. The S.F. is constant and equal to W for all sections between B and A.

The B.M. is zero at B and increases by a straight line law to Wl at the fixed end A.

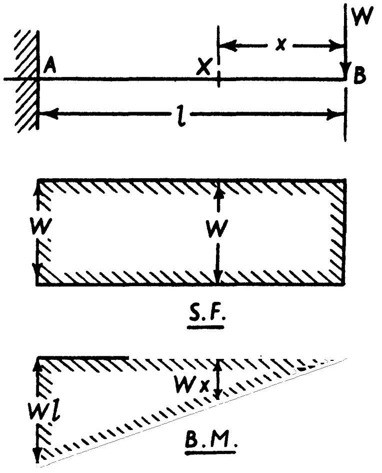


FIG. 37

(ii) Cantilever with a number of point-loads.

A cantilever, 6 feet span, carries loads of $2\frac{1}{2}$, $1\frac{1}{2}$ and 2 tons at 2 feet intervals. Construct the S.F. and B.M. diagrams. For any section X, distant x from D, between D and C,

$$F_x = 2 \text{ tons}$$
$$M_x = 2x \text{ ton-ft.}$$

For a section between C and B, distant x from D,

$$F_x = 2 + 1{\cdot}5 = 3{\cdot}5 \text{ tons}$$
$$M_x = 2x + 1{\cdot}5\ (x-2).$$

For a section between B and A, distant x from D,

$$F_x = 2 + 1{\cdot}5 + 2{\cdot}5 = 6 \text{ tons}$$
$$M_x = 2x + 1{\cdot}5\ (x-2) + 2{\cdot}5\ (x-4).$$

The S.F. diagram is easily drawn, the value of S.F. being constant between consecutive loads. The B.M. between two consecutive loads varies by a straight-line relation. To construct the B.M. diagram, therefore, we should work out the values of the B.M. under each load and at the fixed end, plot these and join up the points by straight lines. Thus,

$$M_D = 0$$
$$M_C = 2 \times 2 = 4 \text{ ton-feet.}$$
$$M_B = 2 \times 4 + 1{\cdot}5 \times 2 = 11 \text{ ton-feet.}$$
$$M_A = 2 \times 6 + 1{\cdot}5 \times 4 + 2{\cdot}5 \times 2 = 23 \text{ ton-feet.}$$

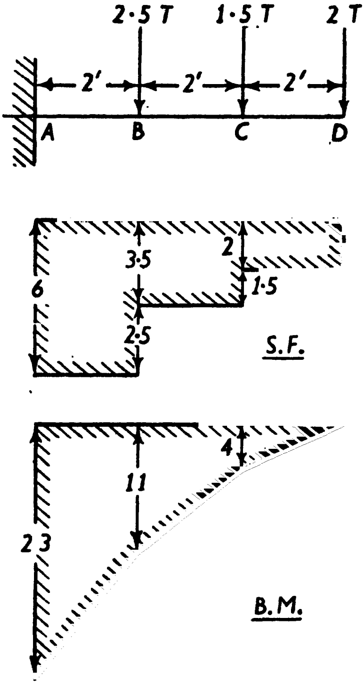


Fig. 38

The maximum S.F. and max. B.M. occur at the fixed end, the values being 6 tons and 23 ton-feet respectively.

(iii) Cantilever of span l, with a uniformly distributed load w per unit length over the entire span.

Consider a section X distant x from the free end.

$$F_x = wx$$

$$M_x = \frac{wx.x}{2} = \frac{wx^2}{2}$$

The S.F. varies by a straight line law and the variation of the B.M. is parabolic. The maxima values are at the fixed end.

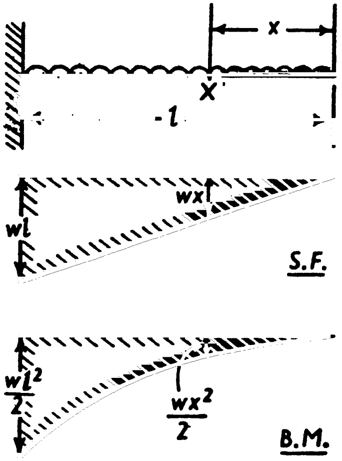


FIG. 39

(iv) Cantilever with partially distributed load.

A cantilever, 6 feet long, carries a uniformly distributed load of $\frac{3}{4}$ ton per foot run over the right hand half. Construct the S.F. and B.M. diagrams.

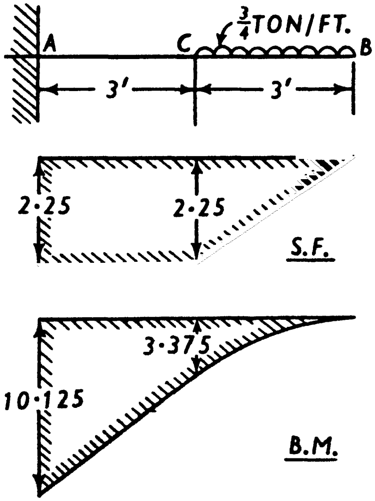


FIG. 40

For a section X in BC, distant x from B,

$$F_x = w.x = \tfrac{3}{4}x \text{ tons}$$

$$M_x = w.x.\frac{x}{2} = \frac{3x^2}{8} \text{ ton-ft.}$$

The laws for S.F. and B.M. for all sections between B and C are respectively straight line and parabolic. At C,

$$F_C = \tfrac{3}{4} \times 3 = 2{\cdot}25 \text{ tons}$$

$$M_C = \tfrac{3}{4} \times 3 \times \tfrac{3}{2} = 3{\cdot}375 \text{ ton-feet.}$$

For any section between C and A, distant x from B,

$$F_x = 3 \times \tfrac{3}{4} = 2{\cdot}25 \text{ tons}$$

$$M_x = 3 \times \tfrac{3}{4}\left(x - \tfrac{3}{2}\right) = \tfrac{9}{4}\left(x - 1{\cdot}5\right) \text{ ton-feet.}$$

The S.F. is constant while the B.M. varies by a straight line law. At A, $F_A = 2{\cdot}25$ tons.

$$M_A = \tfrac{9}{4} \times \tfrac{9}{2} = 10{\cdot}125 \text{ ton-feet.}$$

The S.F. and B.M. diagrams are shown in fig. 40.

(v) Cantilever with combined point-loads and uniformly distributed loads.

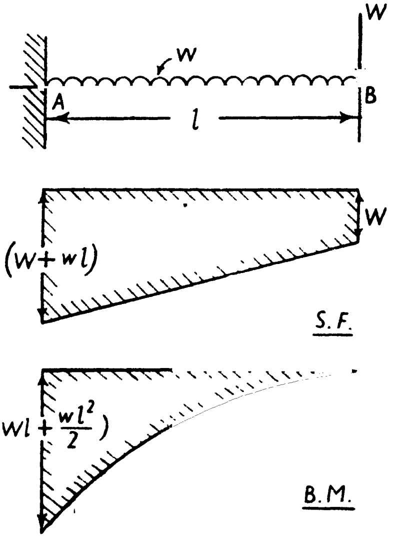


FIG. 41

(a) Uniformly distributed load w per unit length over the entire span and a point load W at the free end. At a distance x from B,

$$F_x = W + wx$$

$$M_x = W.x + wx.\frac{x}{2}$$

$$= W.x + \frac{wx^2}{2}$$

The maxima values at the fixed end A are:

$$F_A = W + wl$$

$$M_A = Wl + \frac{wl^2}{2}.$$

(b) A cantilever, 6 feet span. Point loads of 4000, 3000 and 2000 lb-wt. at 1, 3 and 6 feet from the fixed end and a distributed load of 500 lb. per foot run over two feet from the free end. To construct the S.F. and B.M. diagrams.

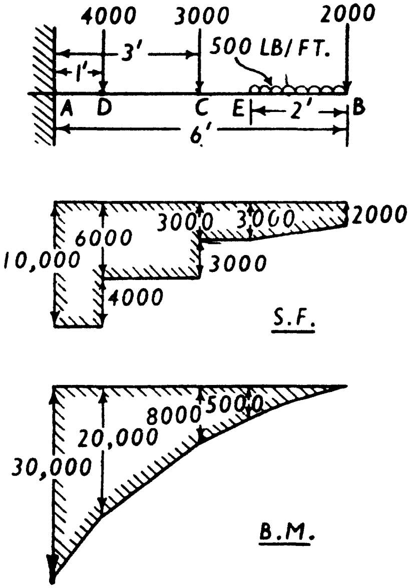


FIG. 42

In the portion BE, at a distance x from B·

$$F_x = 2000 + 500x.$$

$$M_x = 2000x + 500x.\frac{x}{2}$$

$$= 2000x + 250x^2.$$

The S.F. follows a straight line law and the B.M. curve is parabolic. Between *E* and *A*, the S.F. is constant from load to load and the B.M. diagram consists of straight lines.

$M_E = 2000 \times 2 + 250 \times 4 = 5000$ lb-feet.

$M_C = 2000 \times 3 + 1000 \times 2 = 8000$ lb-feet..

$M_D = 2000 \times 5 + 1000 \times 4 + 3000 \times 2 = 20000$ lb-feet.

$M_A = 2000 \times 6 + 1000 \times 5 + 3000 \times 3 + 4000 \times 1$
$= 30000$ lb-feet.

The S.F. and B.M. diagrams are shown in fig. 42.

II. Beams freely supported at ends:

(i) Effective span *l*. Central load *W*.

The reactions at supports,

$$R_A = R_B = \frac{W}{2}.$$

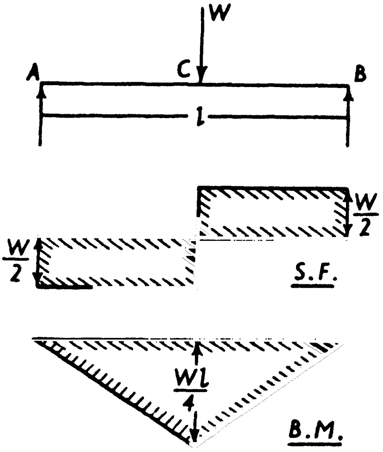


FIG. 43

For a section between *B* and *C*, at a distance *x* from *B*,

$$F_x = R_B = \frac{W}{2}$$

$$M_x = R_B.x = \frac{W}{2}.x.$$

The shear force is a constant between B and C and equal to $\frac{W}{2}$. The Bending Moment is a straight line, reaching a maximum value of $\frac{Wl}{4}$ at C.

In the portion AC, $F_x = \frac{W}{2} - W = -\frac{W}{2}$.

The S.F. is numerically the same, but it changes sign. The diagram is on either side of the base line. This is to be expected, as in the portion AC, the nature of the shear force is ↑ ↓ while in the portion CB, it is ↓ ↑.

The maximum B.M. occurs at the central section. It may be noted that the maximum B.M. occurs at a section where the S.F. is either zero or changes sign.

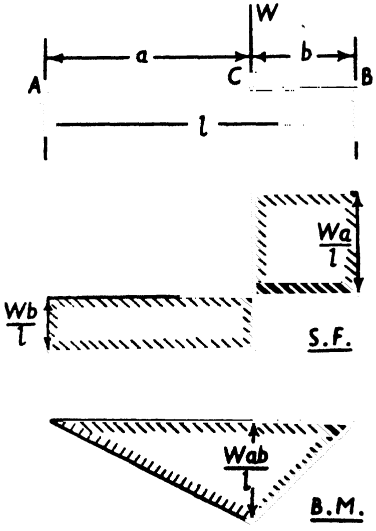


FIG. 44

(ii) A single eccentric load W.

Let the load W be placed at C so that $AC = a$ and $CB = b$, $(a + b)$ being equal to l. The reactions are $R^A = \frac{Wb}{l}$ and $R^B = \frac{Wa}{l}$. For any section in the portion CB, distant x from B,

$$F_x = R_B = \frac{Wa}{l}$$

$$M_x = R_B.x = \frac{Wax}{l}$$

The S.F. is constant and the B.M. diagram is a straight line. As soon as we cross C into the portion CA, the S.F. drops by W, becoming $F_x = R_B - W = \frac{Wa}{l} - W = -\frac{Wb}{l} = -R_A$.

The value of the B.M. at C is,

$M_C = \frac{Wab}{l}$. This is a maximum.

The B.M. diagram consists of straights with zero values at end supports, reaching the maximum of $\frac{Wab}{l}$ at C, where it may be noted that the shear force changes sign.

(iii) Freely supported beam with a number of point-loads.

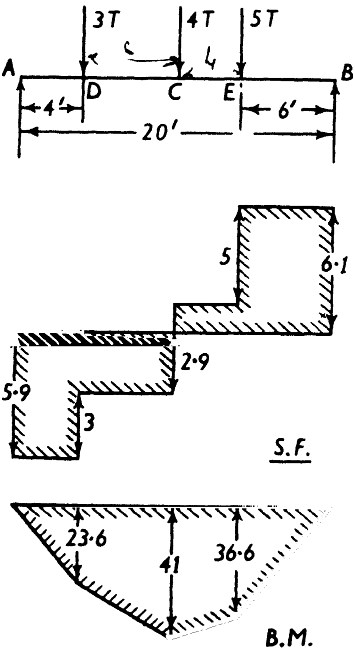


Fig. 45

A beam freely supported over an effective span of 20 feet carries point-loads of 3, 4 and 5 tons at 4, 10 and 14 feet respectively from the left hand support. Construct the S.F. and B.M. diagrams.

The reactions at the supports are 5·9 ton-wt. at *A* and 6·1 ton-wt. at *B*. The Shear Force diagram presents no difficulty. The B.M. follows a straight line law from load to load. Calculate the values of the B.M. at sections under the loads, plot them to scale and join up by straights.

For the B.M. diagram,

$M_E = 6{\cdot}1 \times 6 = 36{\cdot}6$ ton-feet.

$M_C = 6{\cdot}1 \times 10 - 5 \times 4 = 41$ ton-feet.

$M_D = 5{\cdot}9 \times 4 = 23{\cdot}6$ ton-feet.

The maximum B.M. is 41 ton-feet at *C*, where the S.F. changes sign.

(iv) Uniformly distributed load *w* per unit length over the entire span *l*.

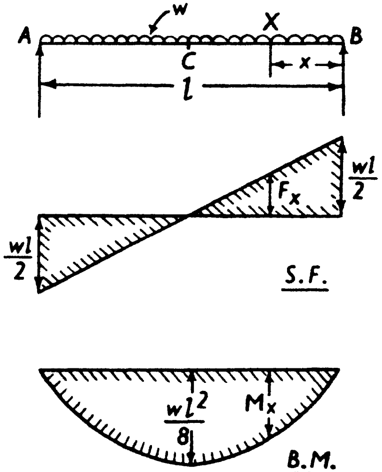


FIG. 46

The total load $W = w.l.$

The reactions at the supports are each $\frac{1}{2}W$ or $\frac{wl}{2}$.

At a section distant x from B,

$$F_x = R_B - wx = \frac{wl}{2} - wx$$

$$M_x = R_B . x - wx . \frac{x}{2}$$

$$= \frac{wlx}{2} - \frac{wx^2}{2} = \frac{wx\ (l-x)}{2}$$

The S.F. diagram is a straight line, being $\frac{wl}{2}$ at B and dropping uniformly by w for every foot as the section moves away from B. At the centre, $F_C = \frac{wl}{2} - \frac{wl}{2} = 0$. Beyond C in the portion CA, it changes sign ending with $\frac{wl}{2}$ at A. The B.M. diagram is a parabola with zero values at the supports, reaching a maximum of $\frac{wl^2}{8}$ or $\frac{Wl}{8}$ at the centre, where the S.F. is zero.

(v) Partially distributed load.

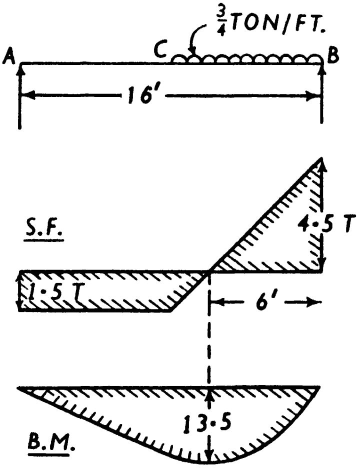


FIG. 47

(a) A freely supported beam of 16 feet effective span carries a load of $\frac{3}{4}$ ton per foot run over the right-hand half of the span. Construct the S.F. and B.M. diagrams.

The reactions at the supports are $R_A = 1{\cdot}5$ tons; $R_B = 4{\cdot}5$ ton-wt. For any section in the portion *CB* distant x from *B*,

$$F_x = R_B - w.x$$

$$= 4{\cdot}5 - \frac{3x}{4},$$ which is a straight line starting with 4·5 tons at *B* and reaching $(4{\cdot}5 - 6)$ or $-1{\cdot}5$ tons at *C*.

The S.F. is zero where $4{\cdot}5 - \frac{3x}{4} = 0$

or $x = 6$ feet from *B*.

$$M_x = 4{\cdot}5x - \frac{3x}{4}.\frac{x}{2}$$

$$= 4{\cdot}5x - \frac{3x^2}{8}.$$

This is an equation of a parabola. The value of the B.M. at *C* is $M_C = 4{\cdot}5 \times 8 - \frac{3}{8} \times 8 \times 8 = 36 - 24 = 12$ ton-feet.

For the maximum, differentiate the expression for M_x with regard to x and equate to zero.

Thus, $\frac{dM_x}{dx} = 4{\cdot}5 - \frac{3}{8} \times 2x = 0$

$\therefore \quad 6x = 36$

or $\quad x = 6$ feet from *B*.

$$M_{\max} = 4{\cdot}5 \times 6 - \frac{3}{8} \times 6 \times 6$$

$= 27 - \frac{27}{2} = 13{\cdot}5$ ton-feet. The maximum B.M. has occured at the section where the S.F. is zero.

We could have obtained the position and amount of maximum Bending Moment by re-arranging the expression for M_x into: $\frac{3}{8}\,[12x - x^2]$

i.e. $\frac{3}{8}\,[36 - (6 - x)^2]$

The value of M_x is always less than $\frac{3}{8} \times 36$ or 13·5 ton-feet at any section, the maximum value being 13·5 ton-feet, when $(6 - x)^2 = 0$ i.e. when $x = 6$ feet.

For any section between *C* and *A*, the S.F. is constant and equal to 1·5 tons, while the B.M. diagram is a straight line. The S.F. and B.M. diagrams are shown in fig. 47.

(b) A freely supported beam, of 16 feet effective span, has a load of 3 tons per foot run uniformly distributed over 2 feet, 3 feet away from the centre towards the right. Construct the S.F. and B.M. diagrams.

The reactions are $R_A = 1{\cdot}5$ tons and $R_B = 4{\cdot}5$ tons.

In the portion BE, for any section distant x from B,

$F_x = 4{\cdot}5$ tons

$M_x = 4{\cdot}5x$

$M_E = 4{\cdot}5 \times 3 = 13{\cdot}5$ ton-feet.

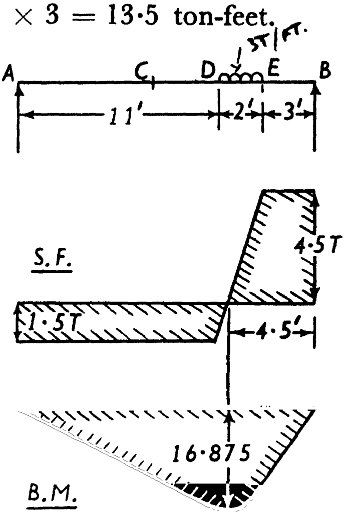


Fig. 48

In the portion ED,

$F_x = 4{\cdot}5 - 3\ (x - 3)$ which is a straight line, becoming zero at $x = 4{\cdot}5$ feet.

At D, $F_D = 4{\cdot}5 - 3 \times 2 = -1{\cdot}5$ tons

$$M_x = 4{\cdot}5x - \frac{3\ (x-3)\ (x-3)}{2}$$

$= 4{\cdot}5x - \frac{3}{2}\ (x-3)^2$, which is a parabola. For the maximum, differentiate the expression with regard to x and equate to zero.

Thus, $\dfrac{dM_x}{dx} = 4{\cdot}5 - 3\ (x - 3) = 0$ or $x = 4{\cdot}5$ feet.

$$M_{\max} = 4{\cdot}5 \times 4{\cdot}5 - \tfrac{3}{2} \times 1{\cdot}5^2$$
$$= 20{\cdot}25 - 3{\cdot}375$$

$= 16{\cdot}875$ ton-feet, at 4½ feet from *B*.

Also, $M_D = 4{\cdot}5 \times 5 - 3 \times 2 = 16{\cdot}5$ ton-feet.

In the portion *AD*, the S.F. is constant at 1½ tons and the B.M. is a straight line. The diagrams may now be drawn as shown in fig. 48.

iv Combined point loads and distributed loads.

A beam is freely supported over a span of 20 feet. It carries a uniformly distributed load of 1 ton per foot run over 5 feet from the left hand support and also from the centre upto the right hand support. It has, besides, two point loads of 2 and 5 tons at 5′ and 15′ from the left hand support. Construct the S.F. and B.M. diagrams.

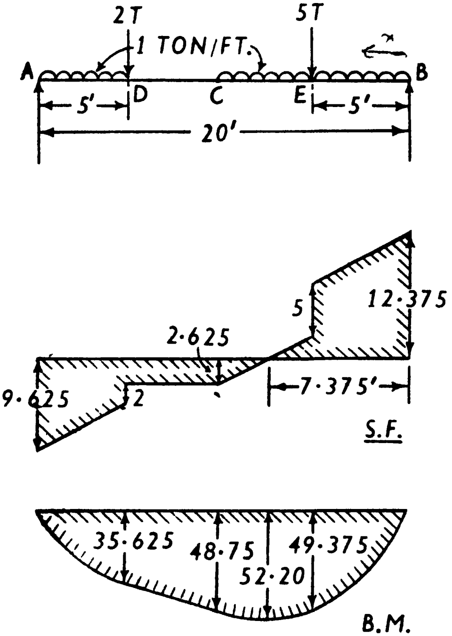


FIG. 49

The reaction at the supports are,

$R_A = 9{\cdot}625$ tons

$R_B = 12{\cdot}375$ tons.

The S.F. diagram may be started from *B* with the upward reaction of 12·375 tons. It presents no difficulty.

It may be noted that the position of the section where the S.F. is zero is 2·375 feet from *E* or 7·375 from *B*.

For any section between *B* and *E*,

$M_x = 12{\cdot}375x - \frac{x^2}{2}$. The curve will be parabolic.

$M_E = 12{\cdot}375 \times 5 - 12{\cdot}5 = 49{\cdot}375$ ton-feet.

Between *E* and *C*,

$M_x = 12{\cdot}375x - \frac{x^2}{2} - 5\ (x - 5)$, which also is parabolic.

At *C*,

$M_C = 12{\cdot}375 \times 10 - 50 - 25 = 48{\cdot}75$ ton-feet.

For the maximum B.M., differentiate M_x with regard to x and equate to zero.

Thus, $\frac{dM_x}{dx} = 12{\cdot}375 - x - 5 = 0$

or, $x = 7{\cdot}375$ feet from *B*.

$$M_{\text{max}} = 12{\cdot}375 \times 7{\cdot}375 - \frac{7{\cdot}375^2}{2} - 5 \times 2{\cdot}375$$

$= 52{\cdot}20$ ton-feet. The maximum occurs at the section where the S.F. is zero.

For any section between *A* and *D*, distant x from *A*

$M_x = 9{\cdot}625x - \frac{x^2}{2}$, which is parabolic.

$M_D = 9{\cdot}625 \times 5 - 12{\cdot}5 = 35{\cdot}625$ ton-feet.

Between *D* and *C*,

$$M_x = 9{\cdot}625x - 5\ (x - \tfrac{5}{2}) - 2\ (x - 5)$$

$= 2{\cdot}625x + 22{\cdot}5$, which is a straight line.

At *C*,

$M_C = 2{\cdot}625 \times 10 + 22{\cdot}5 = 48{\cdot}75$ ton-feet, as already found.

From a study of the S.F. and B.M. diagrams in fig. 49, it will be noticed that where the S.F. is constant, the B.M. is a straight line and where the S.F. is a straight line, the B.M. diagram is parabolic.

III. Overhanging beams:

(i) Beam with overhang at one end. Point-loads.

A beam 20 feet long, has one support at the left-hand end and the other 16 feet away, the beam overhanging it by 4 feet. It carries a load of 1500 lb-wt. at the free end and loads of 2500 and 2000 lb-wt. at 4 and 11 feet respectively from the left support. Construct the S.F. and B.M. diagrams.

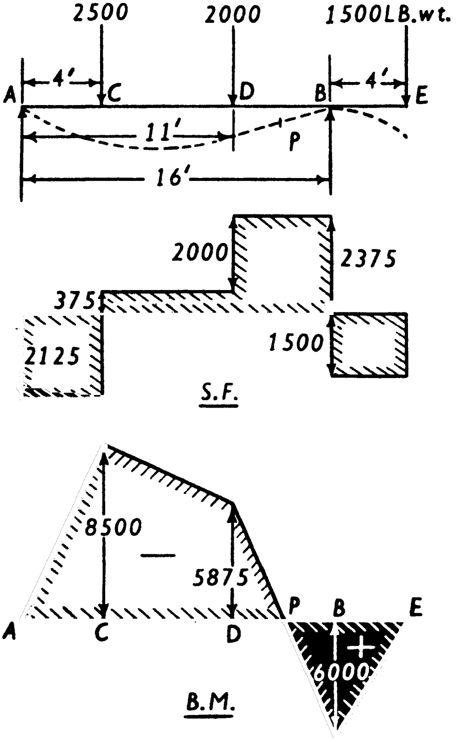


FIG. 50

The reactions at the supports are 2125 lb-wt. at *A* and 3875 lb-wt. at *B*. The S.F. diagram may be drawn without any difficulty, the value of S.F. being constant between loads. It changes sign at *B* and *C*. The B.M. diagram will consist of straights. So we shall work out the values of the B.M. under the loads and at the supports, plot them and complete the diagram. Beginning at *E*,

$$M_E = 0$$
$$M_B = 1500 \times 4 = 6000 \text{ lb-feet clockwise.}$$
$$M_D = 1500 \times 9 - 3875 \times 5 = -19375 + 13500$$
$$= -5875 \text{ lb-feet i.e. } 5875 \text{ lb-feet anticlockwise.}$$

The B.M. has changed sign between *B* and *D*.

$M_C = 2125 \times 4 = 8500$ lb-feet anticlockwise and is of the same sign as M_D. When the B.M. diagram is drawn, it will be on either side of the base line. For all sections from *E* to *P* in fig. 50, the nature of the B.M. is as indicated by the arrows ↺ ↻ and it tends to bend the beam convex upwards, while from *P* to *A*, it is ↻ ↺ tending to produce concavity upwards. The deformation of the beam is indicated by the dotted line in fig. 50. At the section at *P*, the nature of curvature of the beam changes. It is therefore called a point of contrary flexure or "Contra-flexure" or a point of inflexion. It is easy to locate this point, because the B.M. at it is zero.

If the distance of *P* from *E* is *x*,

$$M_p = 1500x - 3875\ (x - 4).$$

Equating this to zero, we get

$$2375x = 15500$$

$\therefore \quad x = 6{\cdot}53$ feet. The point of contraflexure occurs at 6·53 feet from the free end.

In the case of overhanging beams, bending moments will change sign, a part or parts of the beam bending convex upwards and the rest bending concave upwards. For the purpose of drawing the B.M. diagram, we may adopt the convention that bending moments which tend to produce convexity upwards will be considered positive and therefore those tending to produce concavity upwards will be negative. This is only a convention and there is no objection to adopting the reverse. However, the convention that we have suggested is convenient when we come to the study of deflections of beams.

Similarly, for Shear Forces, we shall adopt the convention that upward forces to the right of a section will be considered positive, and downward forces, negative. Therefore upward forces to the left of a section will be treated as negative, the downward forces being positive.

(ii) Beam with overhang at one end. Uniformly distributed load.

A beam 18 feet long is supported at one end, the other support being 15 feet away. It carries a uniformly distributed load of $\frac{1}{2}$ ton per foot run over the entire length. Construct the S.F. and B.M. diagrams.

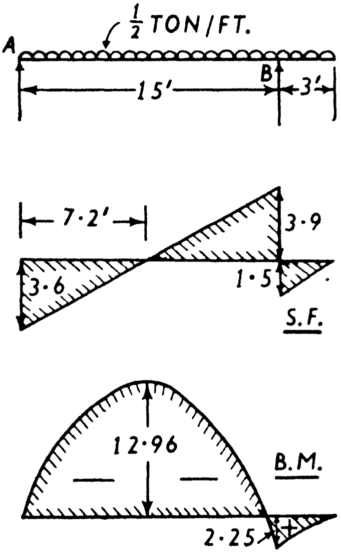


FIG. 51

The reactions at the supports are $R_A = 3{\cdot}6$ tons and $R_B = 5{\cdot}4$ tons. The S.F. diagram is easily drawn. Begin at *C* and proceed towards *A*. Just before crossing the support, the value of *F* is $\frac{1}{2}$ 3 $= 1{\cdot}5$ tons downwards. As soon as we cross *B*, it goes up by 5·4 tons. The diagram is therefore 3·9 tons above the base line at *B*, dropping steadily at $\frac{1}{2}$ ton per foot run until at *A*, F_A is 3·6 tons downwards. The S.F passes through its zero value at 7·2 feet from *A*.

For any section between *C* and *B*, distant *x* from *C*,

$$M_x = w.x.\frac{x}{2} = +\frac{wx^2}{2} = \frac{x^2}{4}$$

This relation is parabolic, the value at B being

$$M_B = \frac{3}{2} \times \frac{3}{2} = 2{\cdot}25 \text{ ton-ft.}$$

This is positive according to our convention.

For any section between B and A, distant x from C,

$$M_x = + w.x.\frac{x}{2} - R_B\ (x-3)$$

$$= + \frac{x^2}{4} - 5{\cdot}4\ (x-3).$$

The effect of R_B being to produce a moment of the opposite nature, the value of M_x starts diminishing from $+2{\cdot}25$ ton-feet at B, until it becomes zero at some point, changes sign, and after reaching the maximum negative value at some section, becomes zero again at A.

For the position of the point of contraflexure equate M_x to zero.

$$\frac{x^2}{4} - 5{\cdot}4\ (x-3) = 0$$

$$\therefore \quad x^2 - 21{\cdot}6x + 64{\cdot}8 = 0$$

$$\text{or} \quad (x-18)\ (x-3{\cdot}6) = 0$$

$$\therefore \quad x = 3{\cdot}6 \text{ feet or } 18 \text{ feet from } C.$$

The B.M. is zero at A and at 3·6 feet from C. This is the point of contraflexure.

For maximum bending moment,

$$\frac{dM_x}{dx} = \frac{2x}{4} - 5{\cdot}4 = 0$$

or $x = 10{\cdot}8$ feet from C. The maximum B.M. occurs at 10·8 feet from C or 7·2 feet from A.

$$M_{\max} = -3{\cdot}6 \times 7{\cdot}2 + \tfrac{1}{2} \times 7{\cdot}2 \times \frac{7{\cdot}2}{2}$$

$$= -3{\cdot}6 \times 3{\cdot}6 = -12{\cdot}96 \text{ ton-feet.}$$

The B.M. diagram may now be completed as in fig. 51.

It will be noted that the maxima values of the B.M. occur at sections where the S.F. is either zero or it changes sign. The maximum negative value of 12·96 ton-feet occurs at 7·2 feet from *A*, where the S.F. is zero and the maximum positive value of 2·25 ton-feet is at *B* where the S.F. changes sign.

(iii) Beam with overhang at one end. Uniformly distributed load over the entire length and a point load at the free end.

A beam of 20 feet length is hinged at one end and freely supported at a distance of 15 feet from the hinge. It carries a uniformly distributed load of $\frac{1}{2}$ ton per foot run over the entire length and a point-load of 16 tons at the free end. Construct the S.F. and B.M. diagrams.

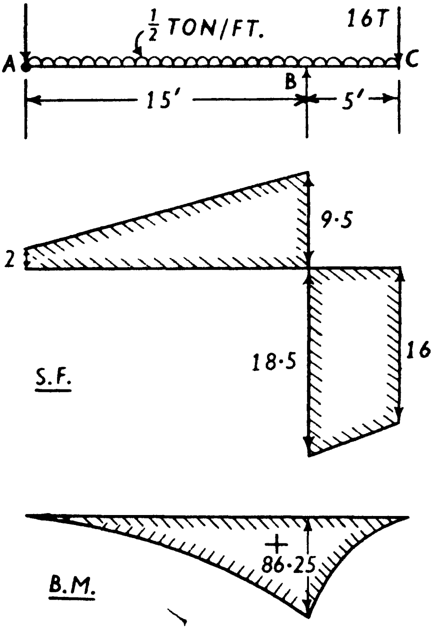


Fig. 52

The reactions at the supports are $R_B = 28$ tons upwards, while R_A at the hinge is 2 tons downwards.

The S.F. diagram begins with 16 tons downwards at *C* and drops at $\frac{1}{2}$ ton per foot run as we proceed towards *B* where it reaches the value of 18·5 tons downwards. As soon as we cross *B*, it goes up by $R_B = 28$ tons upwards, coming above the base line by 9·5 tons. Thence onward to *A* it drops steadily at $\frac{1}{2}$ ton per foot run until at *A* it reaches the value of 2 tons. The shear force does not become zero anywhere but changes sign at the support *B*. For any section between *C* and *B*, distant *x* from *C*,

$$M_x = +16x + \tfrac{1}{2}.x.\frac{x}{2}$$

$$= +16x + \frac{x^2}{4} \text{ reaching the value at } B \text{ of}$$

$$M_B = 16 \times 5 + \frac{25}{4} = +86{\cdot}25 \text{ ton-feet.}$$

For any section between *B* and *A*, the bending moment will start diminishing due to the opposite moment of the reaction R_B.

For a section distant *x* from *A*,

$$M_x = +2x + \tfrac{1}{2}x.\frac{x}{2} = 2x + \frac{x^2}{4}$$

$$\text{At } B,\ M_B = 2 \times 15 + \frac{15^2}{4} = 30 + 56{\cdot}25 = 86{\cdot}25$$

ton-feet as before.

The B.M. is positive throughout and does not change sign. There is no point of contraflexure and the beam will bend convex upwards along its entire length.

(~~i~~v) Overhang on both sides. Point-loads at ends.

A beam of length $(l + 2a)$ with supports l apart, overhang of a at each end, carries a load W at each end. Construct the S.F. and B.M. diagrams.

The reactions at the supports are each *W*. The S.F. between *E* and *B* is *W* downwards. Between *B* and *A*, there is no shear force. Between *A* and *D*, it is again *W* but opposite in sign.

The B.M. at any section between *E* and *B*, distant *x* from *E* is $M_x = +W.x$, a straight line reaching the value $+W.a$ at *B*. Between *B* and *A*, the B.M. is constant and equal to $+W.a$. Between *A* and *D*, the B.M. diagram is the same as in the portion *BE*. The B.M. is throughout positive and the beam will bend convex upwards.

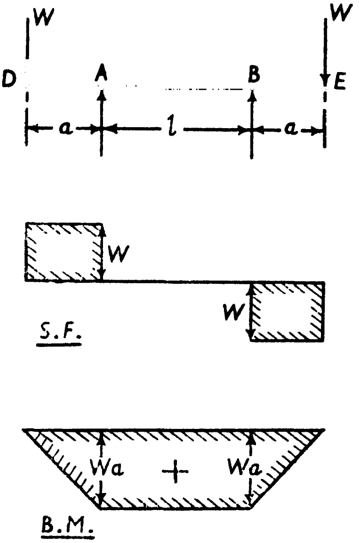


FIG. 53

(v) Overhang on both sides. A number of point loads on the beam.

A beam 20 feet long has supports 14 feet apart, overhanging 2 feet on the left and 4 feet on the right. It carries a load of 4000 lb-wt. at the left end and 3000 lb-wt. at the right, besides three loads of 3000, 2000 and 4000 lb-wt. at 3, 7 and 10 feet from the left support. Construct the S.F. and B.M. diagrams.

The reactions at the supports are $R_A = 8214$ lbs. and $R_B = 7786$ lbs. The S.F. diagram may be drawn as usual, the value keeping steady until a load is encountered, when it suddenly changes by the amount of the load. The S.F. changes sign at the supports and at *C*, the middle point of *AB*. The B.M. diagram will consist of straights. It is therefore enough to calculate the values of the B.M. under each load and at the supports, plot them and join up the straights from load to load to complete the diagram.

$$
\begin{aligned}
M_B &= 3000 \times 4 = +12000 \text{ lb-feet}\\
M_G &= +3000 \times 8 - 7786 \times 4\\
&= +24000 - 31144 = -7144 \text{ lb-feet.}\\
M_C &= +3000 \times 11 - 7786 \times 7 + 4000 \times 3\\
&= +33000 - 54502 + 12000 = -9502 \text{ lb-feet.}\\
M_F &= +4000 \times 5 - 8214 \times 3, \text{ working from } D\\
&= +20000 - 24642 = -4642 \text{ lb-feet.}\\
M_A &= +4000 \times 2 = +8000 \text{ lb-feet.}
\end{aligned}
$$

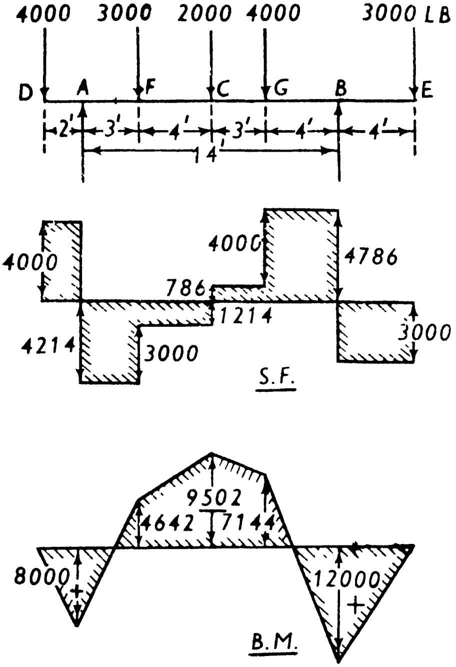


Fig. 54

The B.M. diagram may be completed as in fig. 54. The maximum positive B.M. is at *B* and the maximum negative at *C*. At both places, the shear force changes sign. There is a point of contraflexure between *A* and *F* and another between *G* and *B*.

Let the distance of the first one be x from *D*. Then

$$4000x - 8214\ (x - 2) = 0$$

$$\therefore \quad x = 3{\cdot}894 \text{ feet from } D.$$

If the distance of the other point of inflexion is x from E,

$$3000x - 7786\,(x-4) = 0$$

$$\text{i.e.} \quad 4786x = 31144$$

$$\therefore \quad x = 6{\cdot}5 \text{ feet from } E.$$

From the ends to the points of inflexion, the beam bends convex upwards and it will be concave upwards between the points of inflexion.

(vi) Beam with double overhang. Uniformly distributed load.

(a) A beam 20 feet long has supports 12 feet apart with an equal overhang of 4 feet at each end. The beam has a uniformly distributed load of $\frac{1}{2}$ ton per foot run over the entire length. Construct the S.F. and B.M. diagrams.

The reactions at the supports are 5 tons each. Drawing the S.F. diagram is a straightforward job. The B.M. at E is,

$$M_E = +\tfrac{1}{2} \times 4 \times \tfrac{4}{2} = +4 \text{ ton-feet.}$$

At the centre C, the B.M. is,

$$M_C = +\tfrac{1}{2} \times 10 \times 5 - 5 \times 6 = +25 - 30$$
$$= -5 \text{ ton-feet.}$$

The S.F. and B.M. diagrams are shown in fig. 55.

For points of contraflexure, if x is the distance from B,

$$M_x = +\tfrac{1}{2}.\, x.\frac{x}{2} - 5\,(x-4)$$

$$= \frac{x^2}{4} - 5\,(x-4) = 0$$

$$\therefore \quad x^2 - 20x + 80 = 0$$

$$\therefore \quad (x-10)^2 = 20$$

$$\therefore \quad x = 10 \pm \sqrt{20}$$

$$= 10 \pm 4{\cdot}472 \text{ i.e. } 5{\cdot}528 \text{ or } 14{\cdot}472 \text{ feet from } B.$$

The distance of the point of contraflexure is $\sqrt{20}$ or 4·472 feet on either side of the centre C.

The maximum positive moment is at either support. The maximum negative moment is at the centre. Of the two, which will be the greater depends upon the position of the supports. In this problem, the central negative moment is the larger. If the supports are moved nearer the centre, the support-moment will increase and the central moment will diminish. There is one particular position of the supports for which the support-moment is numerically equal to the central moment. In this case, the maximum moment, positive or negative, is as small as possible.

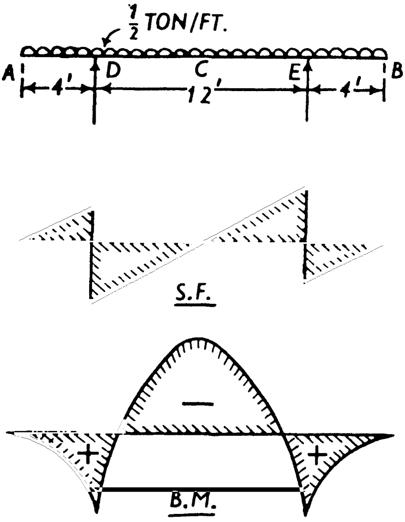


FIG. 55

Consider a beam of length *l* carrying a uniformly distributed load of *w* per unit length. To find the position of the supports so that the maximum B.M. is as small as possible.

Let the supports be placed at a distance of x from either end. The reactions at the supports will be $\frac{wl}{2}$ each.

The support-moment,

$$M_E = + w.x.\frac{x}{2} = + \frac{wx^2}{2}.$$

The central moment,

$$M_C = +\frac{wl}{2}\cdot\frac{l}{4} - \frac{wl}{2}\left(\frac{l}{2} - x\right).$$

This is negative. If the support moment is to be numerically equal to the central moment, the algebraic sum of M_E and M_C must be zero.

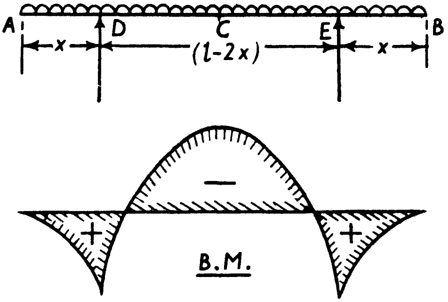


FIG. 56

$$\frac{wx^2}{2} + \frac{wl^2}{8} - \frac{wl}{2}\left(\frac{l}{2} - x\right) = 0$$

or $$x^2 + \frac{l^2}{4} - \frac{l^2}{2} + lx = 0$$

∴ $$x^2 + l.x = \frac{l^2}{4}$$

∴ $$\left(x + \frac{l}{2}\right)^2 = \frac{l^2}{2}$$

∴ $$\left(x + \frac{l}{2}\right) = \pm\frac{l}{\sqrt{2}}$$

∴ $$x = \frac{l}{\sqrt{2}} - \frac{l}{2} = \frac{l}{2}(\sqrt{2} - 1) = \frac{\cdot 414l}{2} = \cdot 207l.$$

If the supports are arranged at $\cdot 207l$ from either end, the maximum B.M. positive or negative, is as small as possible.

If the overhang is large as compared with the supported length, the B.M. in the supported portion may not change sign at all, the B.M. diagram being as shown in fig. 57. In this case, the beam will bend convex upwards throughout.

(b) Beam with double overhang. Point-loads and uniformly distributed loads.

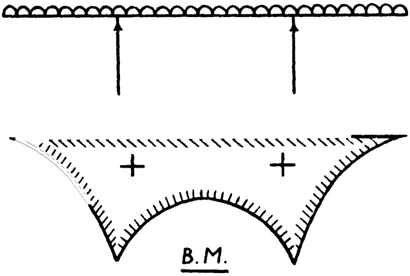


FIG. 57

A beam 30 feet long, has supports 20 feet apart, there being an overhang of 4 feet on the left and 6 feet on the right. There

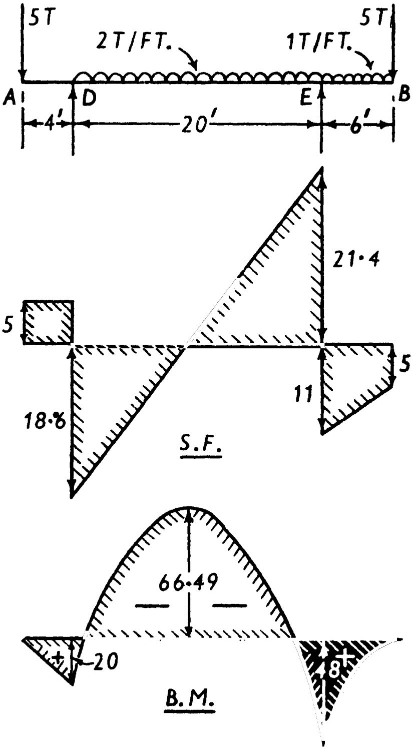


FIG. 58

is a point load of 5 tons at each free end, a uniformly distributed load of 2 tons per foot run over the supported length and 1 ton per foot run over the overhanging portion on the right. Construct the S.F. and B.M. diagrams.

The reactions at the supports are $R_D = 23{\cdot}6$ tons and $R_E = 32{\cdot}4$ tons. The S.F. and B.M. diagrams are shown in fig. 58. The B.M. diagram is a straight line in the portion *AD* and is parabolic from *D* to *E* and from *E* to *B*. The support moments are

$$M_D = 5 \times 4 = +20 \text{ ton-feet}$$

and $M_E = 5 \times 6 + 6 \times 3 = +48$ ton-feet.

The maximum negative moment which is 66·49 ton-feet occurs where the S.F. is zero, i.e. 10·7 feet from *E*. The points of contraflexure are 1·146 feet from *D* and 2·546 feet from *E*.

4. Relation between the S.F. and the B.M. at a cross-section of a beam: We have drawn the S.F. and B.M. diagrams for cantilevers and beams, freely supported or overhanging at ends, carrying point-loads and uniformly distributed loads. We have noticed that when the S.F. is constant, the B.M. diagram is a straight line and when the S.F. diagram is a straight line, the B.M. diagram is a parabolic curve. Mathematically it would appear that the curve of the B.M. diagram is one degree higher than the corresponding S.F. curve. We have also noticed that the maximum B.M. occurs at sections where the S.F. is either zero or changes sign. That each of the

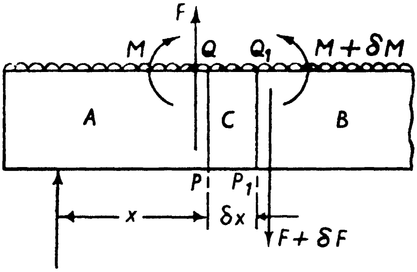


FIG. 59

actions, viz., the S.F. and the B.M. at a cross-section, must be a function of the loading and the span of the beam, is evident. But there must be a definite relation between these two actions, which we shall now investigate.

Consider the equilibrium of a portion C of the beam of length δx, at a distance x from the left hand support. The forces acting on it consist of:

(i) The load $w.\delta x$ on it where w is the rate of loading on the beam, uniform or variable.

(ii) The actions exerted by the portion A of the beam at the interface PQ. These are a shear force F and a Bending Moment M.

(iii) The actions exerted by the portion B of the beam at the interface P_1Q_1. These consist of a shear force $F + \delta F$, and a Bending Moment $M + \delta M$.

These forces and couples acting on the portion C of the beam are marked in fig. 59. Since C is in equilibrium, the system of forces acting on it obeys the laws of equilibrium.

Resolving vertically,

$$F + \delta F + w.\delta x = F$$

i.e. $\delta F + w.\delta x = 0$ .... (1)

Taking moments of the forces and couples about the section P_1Q_1, $M + \delta M = M + F.\delta x - w.\delta x.\ \frac{\delta x}{n}$, the value of n being 2 if the rate of loading is uniform.

$$\therefore \quad \delta M = F.\delta x - \frac{1}{n}.w.\delta x^2$$

$$= F.\delta x \quad \text{.................................} \quad (2)$$

neglecting small quantities of higher order.

We, therefore, deduce that,

$$\frac{dF}{dx} = -w$$

and $\frac{dM}{dx} = F$, i.e., that (i) the rate of change of S.F. at a cross-section is equal to the rate of loading at the section, and (ii) the rate of change of Bending Moment is equal to the shear force at the cross-section.

If the rate of loading at a section is zero, as in the case of point loads on a beam,

$$\frac{dF}{dx} = 0$$

or $F =$ a constant.

In this case, $\dfrac{dM}{dx} = F =$ a constant.

$$\therefore \quad \delta M = F.\delta x$$

$$\therefore \quad \Sigma\delta M = \Sigma F.\delta x$$

$\therefore \quad M = F.x + C$, which is a straight line.

If the beam has a uniformly distributed load w on it,

$$\frac{dF}{dx} = -w.$$

$$\therefore \quad \delta F = -w.\delta x.$$

Integrating, $F = -wx + C_1$, where C_1 is a constant.

The shear force, therefore, follows a straight line law. In this case,

$$\frac{dM}{dx} = F = -wx + C_1$$

$$\therefore \quad \delta M = (-wx + C_1)\,\delta x.$$

Integrating,

$M = -\dfrac{wx^2}{2} + C_1x + C_2$, where C_2 is a constant.

The B.M. curve is therefore a parabola, as we have seen in so many S.F. and B.M. diagrams, where the loading was uniformly distributed. We also see now, why the B.M. is a maximum where S.F. is zero.

For a maximum value of M

$$\frac{dM}{dx} = 0;$$

but $\dfrac{dM}{dx} = F$

$$\therefore \quad F = 0.$$

5. S.F. and B.M. diagrams for beams with variable loading: (i) Beam of effective span l carrying a triangular load, zero per foot run at the ends, increasing uniformly to w per foot run at the centre. Construct the S.F. and B.M. diagrams.

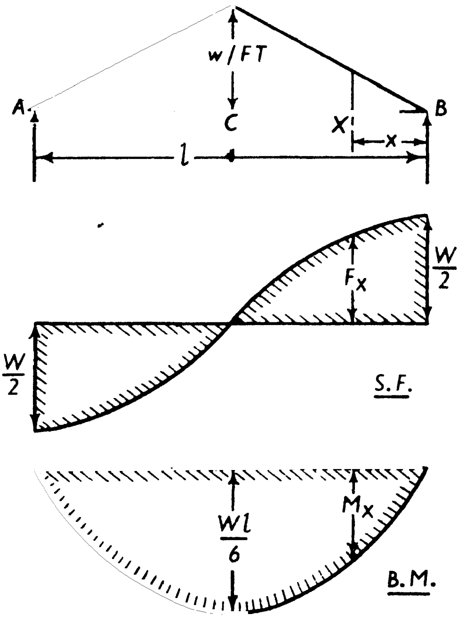


FIG. 60

Total load = average rate of loading span

i.e. $W = \frac{w}{2}.l.$

The reactions at the supports are,

$$R_A = R_B = \frac{W}{2} = \frac{wl}{4}$$

At a section X, distant x from B, the rate of loading is

$$w_x = w \times \frac{x}{l/2} = \frac{2wx}{l}$$

The load over the portion BX of the beam is,

$W_x =$ average rate $\times x = \frac{1}{2} \times \frac{2wx}{l} \times x = \frac{wx^2}{l}$, the distance of the centre of gravity of the load being $\frac{x}{3}$ from the section X.

$$F_x = \frac{W}{2} - W_x = \frac{wl}{4} - \frac{wx^2}{l}, \text{ for all sections from } B \text{ to } C.$$

The S.F. diagram is a parabola, passing through zero at the centre.

$$M_x = -\frac{W}{2}.x + W_x.\frac{x}{3}$$

$$= -\frac{wlx}{4} + \frac{wx^3}{3l}$$

The equation of B.M. diagram is a cubic, one degree higher than that of the S.F. curve. According to our convention, the B.M. is negative throughout, i.e. the beam will bend concave upwards. The maximum B.M. at the centre is,

$$M_C = -\frac{W}{2}\cdot\frac{l}{2} + \frac{W}{2}\cdot\frac{l}{6}$$

$$= -\frac{Wl}{6}$$

It may be noted that when a load W is applied as a concentrated load at the centre of a beam, the maximum B.M. is $\frac{Wl}{4}$; when it is uniformly distributed over the entire span, it is $\frac{Wl}{8}$; when it is piled up as a triangular load with the maximum intensity at the centre, the value of the maximum B.M. is $\frac{Wl}{6}$.

Example 1.

A lintel spanning an opening of 8 feet, carries a brick wall 18" thick. If brick masonry weighs 120 lb. per cubic foot, calculate the maximum B.M. on the assumption that the effective load on the lintel is an equilateral triangle on the span as base. Sketch the S.F. and B.M. diagrams.

The central height of the triangle is,
$h = 4 \tan 60° = 4\sqrt{3}$ feet.

The volume of the effective wall-load on the lintel is,
$V = \frac{1}{2} \times l \times h \times t = \frac{1}{2} \times 8 \times 4\sqrt{3} \times \frac{3}{2} = 24\sqrt{3}$ cub. feet.

The total weight $W = \rho . V = 120 \times 24\sqrt{3}$
$= 4987$, say, 5000 lb-wt.

$$\text{The maximum B.M. at the centre} = \frac{Wl}{6}$$

$$= \frac{5000 \times 8}{6} = 6667 \text{ lb-feet.}$$

At a cross-section distant x from the right hand reaction, the height of wall $h_x = \frac{2hx}{l} = \sqrt{3}\, x$ feet.

The weight of wall over the portion BX (fig. 60) is,
$W_x = \rho . V_x = 120 \times \frac{1}{2}x \times \sqrt{3}x \times t = 90\sqrt{3}.x^2$ lb-wt.

$$F_x = \frac{W}{2} - W_x = 2500 - 90\sqrt{3}\cdot x^2$$

$$M_x = -\frac{W}{2}.x + W_x.\frac{x}{3} = -2500x + 30\sqrt{3}.x^3.$$

The S.F. and B.M. diagrams may now be drawn.

(ii) A beam of effective span l carrying a triangular load which is zero at one end increasing uniformly to w per unit length at the other end.

The total load on the beam is,

$$W = \text{ave. rate} \times \text{span} = \frac{wl}{2}.$$

The reactions at the supports will be

$$R_A = \frac{W}{3} \text{ and}$$

$$R_B = \frac{2W}{3}.$$

At a section X, distant x from A, the rate of loading $w_x = \frac{w.x}{l}$.

The load on the portion AX of the beam is,

$$W_x = \tfrac{1}{2}\, w_x.x = \frac{wx^2}{2l},\ \text{acting at } \frac{x}{3} \text{ from}$$

$$F_x = -R_A + W_x$$

$$= -\frac{wl}{6} + \frac{wx^2}{2l}$$

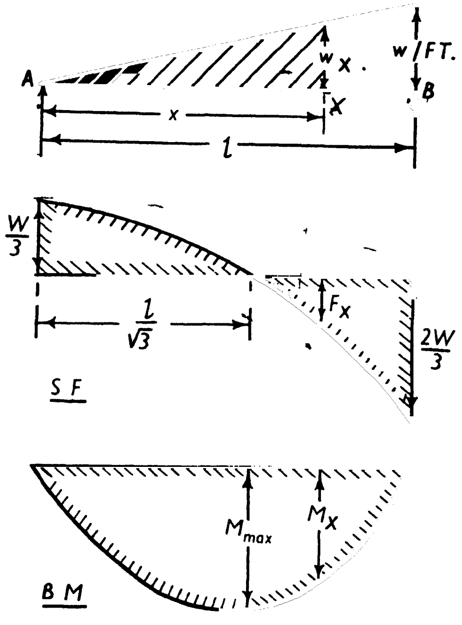


FIG. 61

The S.F. diagram is a parabola, passing through zero at

$$x = \frac{l}{\sqrt{3}}$$

$$M_x = -R_A.x + W_x.\frac{x}{3} = -\frac{wlx}{6} + \frac{wx^3}{6l}$$

The B.M. curve is a cubic, the maximum value of B.M. being at $x = \frac{l}{\sqrt{3}}$

$$M_{\max} = -\frac{wl^2}{6\sqrt{3}}(1-\tfrac{1}{3}) = -\frac{wl^2}{9\sqrt{3}}$$

$$= -\frac{2Wl}{9\sqrt{3}} = -\cdot 128W.l.$$

Example 2.

A beam of effective span 16 feet carries a triangular concrete wall 6″ thick, the height being zero at one end, increasing uniformly to 8 feet at the other. If the weight of concrete is 130 lb. per cubic foot, calculate the maximum B.M. on the beam. Sketch the S.F. and B.M. diagrams.

The volume of the wall-load is,

$V = \frac{1}{2}.l.h.t = \frac{1}{2} \times 16 \times 8 \times \frac{1}{2} = 32$ cub-feet.

The total load $W = \rho.V = 130 \times 32 = 4160$ lb-wt.

The reactions at the supports will be,

$R_A = 1387$ lb-wt. and $R_B = 2773$ lb-wt.

At a section X distant x from the left hand end A, the height of wall $h_x = \frac{hx}{l} = \frac{x}{2}$ feet.

The weight of wall on the portion AX is,

$$W_x = 130 \times \tfrac{1}{2} \times x \times \frac{x}{2} \times t = \frac{65x^2}{4} \text{ lb-wt.}$$

$$F_x = -R_A + W_x = -1387 + \frac{65x^2}{4}$$

$$M_x = -R_A.x + W_x.\frac{x}{3} = -1387x + \frac{65x^3}{12}.$$

The maximum B.M. occurs where

$$-1387 + \frac{65x^2}{4} = 0$$

i.e. at $x = 9{\cdot}24$ feet

$$M_{\max} = -8545 \text{ lb-feet.}$$

The values of F_x and M_x at 2 feet intervals have been tabulated below:

x feet	0	2	4	6	8	10	12	14	16
F_x lb-wt.	-1387	-1322	-1127	-802	-347	$+238$	$+953$	$+1798$	$+2773$
M_x lb-wt.	0	-2731	-5201	-7152	-8323	-8453	-7284	-4554	0

The S.F. and B.M. diagrams may now be plotted.

Example 3.

A bulkhead 24 feet deep, is stiffened by vertical beams, spaced 3 feet centre to centre. If water reaches the top of the bulkhead, calculate the maximum B.M. on a vertical beam. Sketch the S.F. and B.M. diagrams. Density of water = 62·5 lb. per cubic foot.

The intensity of water pressure at the top is zero. At 24 feet below the free surface it is $w.h = 62{\cdot}5 \times 24 = 1500$ lb. per sq. foot.

Since each vertical beam is in charge of 3 feet width of plate of the bulk-head, the horizontal loading on a vertical beam will be zero per foot run at the top, increasing uniformly to 3 × 1500 i.e. 4500 lb. per foot run at the bottom.

The total horizontal thrust of water on the beam is, $W = \frac{1}{2} \times 4500 \times 24 = 54000$ lb-wt. The reactions at the supports will be 18000 lb-wt. at the top and 36000 lb-wt. at the bottom.

At a depth of x ft. from the top, the rate of horizontal water thrust per foot length of the vertical beam is $4500 \times \frac{x}{24}$

i.e. $\frac{1500x}{8}$ lb-wt. per foot run.

$$W_x = \tfrac{1}{2} \times \frac{1500x}{8} \times x = \frac{375x^2}{4} \text{ lb-wt.}$$

$$F_x = -18000 + \frac{375x^2}{4}.$$

$$M_x = -18000x + \frac{125x^3}{4}.$$

The maximum B.M. occurs where

$$-18000 + \frac{375x^2}{4} = 0$$

i.e. at $x = 8\sqrt{3}$ or 13·86 feet from the top.

$$M_{\text{max}} = -18000x + \frac{125x^3}{4}$$

$$= -8\sqrt{3}\ \{18000 - 6000\} \text{ by putting } x = 8\sqrt{3}$$

$$= -96000\sqrt{3} = -166272 \text{ lb-feet.}$$

The values of F_x and M_x at 2 feet intervals are tabulated so that the S.F. and B.M. diagrams can be completed.

x feet	0	2	4	6	8	10	12	14	16	18	20	22	24
F_x lb-wt	−18000	−17675	−16500	−14625	−12000	−8625	−4500	+375	+6000	+12375	+19500	+27375	+36000
M_x lb-feet	0	−35750	−70000	−101250	−128000	−148750	−162000	−166250	−160000	−141750	−110000	−63250	0

Example 4.

The intensity of loading on a beam of 18 feet span increases gradually from 500 lb. per foot run at one end to 1000 lb. per foot run at the other. Find the position and amount of maximum B.M. Construct the S.F. and B.M. diagrams.

The load may be divided into (i) a uniformly distributed load of 500 lb. per foot run throughout the span and (ii) a triangular load which is zero at one end *A*, increasing uniformly to 500 lb. per foot run at the other end *B*.

The end-reactions to (i) are 4500 lb-wt. at each end. The triangular load $= \frac{1}{2} \times 500 \times 18$ i.e. 4500 lb-wt. The reactions to this load are $1/3 \times 4500$ i.e. 1500 lb-wt. at *A* and $2/3 \times 4500$ i.e. 3000 lb-wt. at *B*. The total reactions at *A* and *B* are therefore,

$$R_A = 4500 + 1500 = 6000 \text{ lb-wt.}$$

$$R_B = 4500 + 3000 = 7500 \text{ lb-wt.}$$

Consider a section X, distant x feet from A.

The rate of loading w_x at the section is $\left(500 + \frac{500x}{18}\right)$ lb-wt. per foot run.

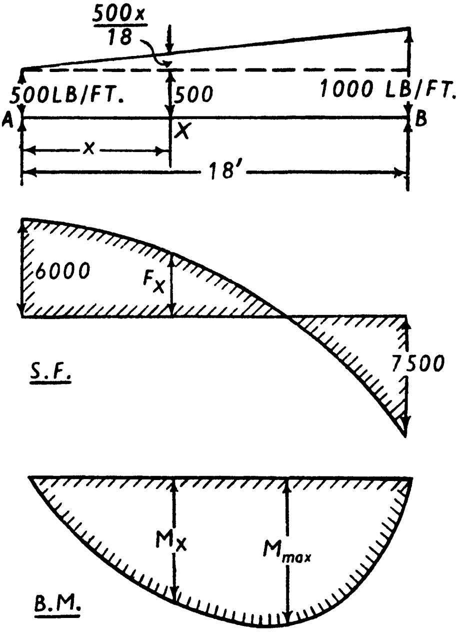


FIG. 62

The load over the portion AX of the beam is,

$$W_x = 500x + \tfrac{1}{2}\left(\frac{500x}{18}\right) \times x$$

$$= 500x + \frac{250x^2}{18}$$

$$\therefore \quad F_x = -R_A + W_x = -6000 + 500x + \frac{250x^2}{18}$$

$$M_x = -6000x + 500x.\frac{x}{2} + \frac{250x^2}{18}.\frac{x}{3}$$

$$= -6000x + 250x^2 + \frac{250x^3}{54}$$

For maximum B.M.,

$$F_x = 0$$

$$\therefore \quad \frac{250x^2}{18} + 500x - 6000 = 0$$

$$\therefore \quad x^2 + 36x = 432$$

i.e. $$(x + 18)^2 = 756$$

$$\therefore \quad x + 18 = 27{\cdot}5$$

or $$x = 9{\cdot}5 \text{ feet.}$$

$$M_{\max} = -6000 \times 9{\cdot}5 + 250 \times 9{\cdot}5^2 + \frac{250}{54} \times 9{\cdot}5^3$$

$$= -30471 \text{ lb-feet.}$$

The values of S.F. and B.M. have been calculated at 3 feet intervals so that the S.F. and B.M. diagrams can be drawn.

x feet	0	3	6	9	12	15	18
Fx lb-wt.	−6000	−4375	−2500	−375	+2000	+4625	+7500
Mx lb-ft.	0	−15625	−26000	−30375	−28000	−18125	0

6. Graphical Methods: The use of graphical methods for the construction of S.F. and B.M. diagrams is to be deprecated. With ordinary loading, these curves are straight lines or parabolas and it is a waste of time employing the usual graphical methods to draw them. The object of the diagrams is essentially to locate sections having the maximum S.F. and maximum B.M. These are easily located and the maxima values of the S.F. and B.M. can be calculated, in most cases, without any difficulty. We shall, however, indicate a graphical method for the S.F. and B.M. diagrams for a freely supported beam with a number of point-loads. The method is very useful for working out the maximum S.F. and maximum B.M. at any section of a girder with travelling point-loads like axle-loads of locomotives.

Consider a beam PQ carrying concentrated loads W_1, W_2 and W_3 as shown in fig. 63. Let it be drawn to a scale of 1″ equal to p feet of its length.

To obtain the reactions at the supports graphically, we construct the funicular polygon. Set off the load-line *abcd* to represent the loads W_1, W_2 and W_3 to a scale of 1 inch $= q$ lb-wt. Select a suitable pole *o*, at a horizontal distance of *h* inches from the load-line. Join *oa*, *ob*, *oc* and *od*. On the line of action of the load W_1, select a suitable point r_1 and through it draw a line parallel to *oa* to cut the line of action of the reaction R_P at p_1 and another line parallel to *ob* to cut the line of action of W_2 as s_1. At s_1 draw s_1t_1 parallel to *oc* to cut the line of action of W_3 at t_1. From t_1, draw t_1q_1 parallel to *od* to cut the line of action of the reaction R_Q at q_1. Join p_1q_1 which is the closing line of the funicular polygon just constructed.

From the pole *o*, draw *oe* parallel to the closing line to cut the load line *abcd* at *e*. Then, *de* and *ea* represent the reactions R_Q and R_P. Project lines through *a*, *b*, *c*, *d* and *e* horizontally. On p' q' (the projection of PQ at level of *e*) as base, the horizontal projection lines through *a*, *b*, *c* and *d* together with the lines of actions of the loads and reactions, give the S.F. diagram for the beam, as shown by the shaded portion in fig. 63.

As for the B.M. diagram, it has already been drawn in as much as the funicular polygon is the B.M. diagram for the beam for the given load system, the vertical intercept of the polygon at any section representing the B.M. at the section to some scale.

Consider a section X, distant x inches from P in fig. 63. Let the vertical intercepts at this section in the S.F. and B.M. diagrams be $x'_1x'_2$ and x_1x_2 respectively. Let z_1 inches be the distance of the line of action of W_1 from X in the diagram. The shear force at the section X is,

$$
\begin{aligned}
F_x &= R_P - W_1,\ \text{numerically} \\
&= q.ea - q.ab = q\ (ea - ab) \\
&= q.x'_1x'_2.
\end{aligned}
$$

The shear force F_x at the section is thus given by $x'_1x'_2$ to a scale of $1'' = q$ lb-wt.

The B.M. at the section X is,

$M_x = R_P \times p.x - W_1.p.z_1$ since the beam is drawn to a scale of $1'' = p$ feet.

$\therefore$ $M_x = q.ea.\ p.x - q.ab.p.z_1$, substituting for R_P and W_1 from the load-line of the force polygon.

$\therefore$ $M_x = p.q\ (ea.x - ab.z_1)$.

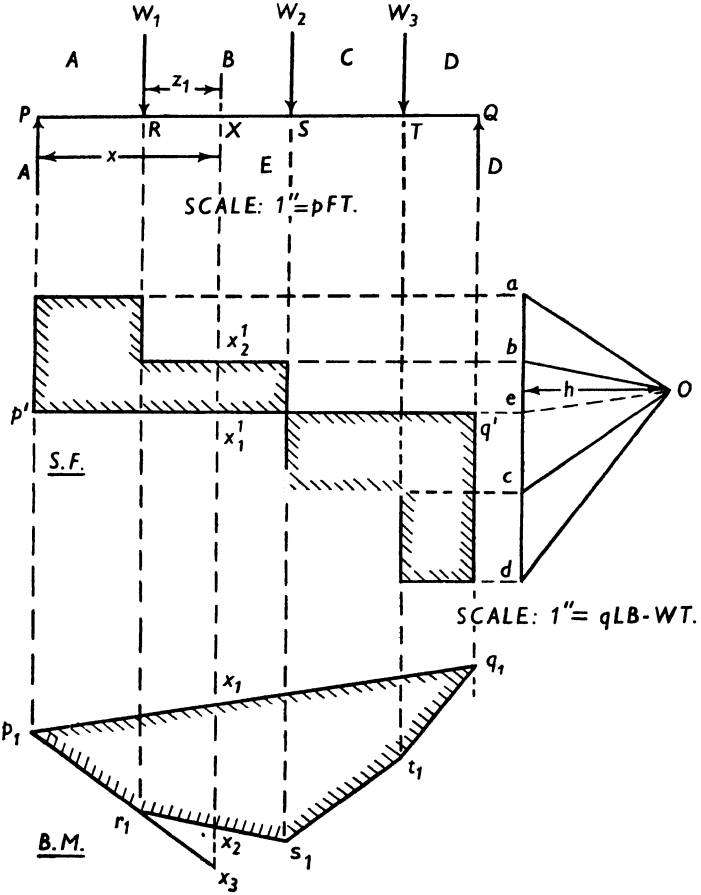


FIG. 63

We shall prove that the expression within the brackets is equal to $h.x_1x_2$ where x_1x_2 is the vertical intercept of the funicular polygon at the section; thus we get $M_x = p.q.h.$ x_1x_2; x_1x_2 will therefore represent the B.M. at X to the scale of 1 inch $= p.q.h$ lb-feet of moment.

Produce p_1r_1 to meet x_1x_2 at x_3. Consider the triangles $p_1x_1x_3$ and oae. They are similar.

$$\therefore \quad \frac{ea}{x_1x_3} = \frac{oe}{x_1p_1} = \frac{\text{horiz. projection of } oe}{\text{horiz. projection of } x_1p_1} = \frac{h}{x}.$$

Cross multiplying,

$$ea.x = h.x_1x_3 \quad \ldots\ldots (1)$$

Likewise, triangles $r_1x_2x_3$ and oab are similar.

$$\therefore \quad \frac{ab}{x_2x_3} = \frac{ob}{r_1x_2} = \frac{\text{horiz. projection of } ob}{\text{horiz. projection of } r_1x_2} = \frac{h}{z_1}$$

Cross multiplying, $ab.z_1 = h.x_2x_3$ ......(2)

Subtracting (2) from (1), we get,

$$(ea.x - ab.z_1) = h\ (x_1x_3 - x_2x_3) = h.x_1x_2,$$

which proves the proposition.

The funicular polygon thus represents the B.M. diagram. If it is desired to have the B.M. diagram on a horizontal base, the closing line of the funicular polygon has to be horizontal. This is achieved by selecting any pole o on the horizontal through the point e for constructing the funicular polygon. If the loading is in ton-units, q will represent the scale of loading in tons.

7. Inclined Beams and beams with oblique loading: If the load system is not at right angles to the axis of the beam, it can be resolved axially and at right angles. The transverse components will produce Shear Forces and Bending Moments while the axial components will give direct pulls or pushes. The cross-section of a beam with inclined loading on it, will have therefore, three actions to resist, viz., (i) a direct force P which may be tensile or compressive, (ii) a shear force F and (iii) a bending moment M. We shall therefore have to draw three diagrams to represent the actions P, F and M across all sections of the beam from one end to the other.

Consider a beam of length L feet weighing w lb. per foot run, vertically supported at A and B, the tops of the supports being l feet horizontally, and h feet vertically apart. If $\theta°$ is the inclination of the beam to the horizontal, $\sin\theta = \frac{h}{L}$ and $\cos\theta = \frac{l}{L}$.

The vertical reactions at A and B are each $\frac{wL}{2}$.

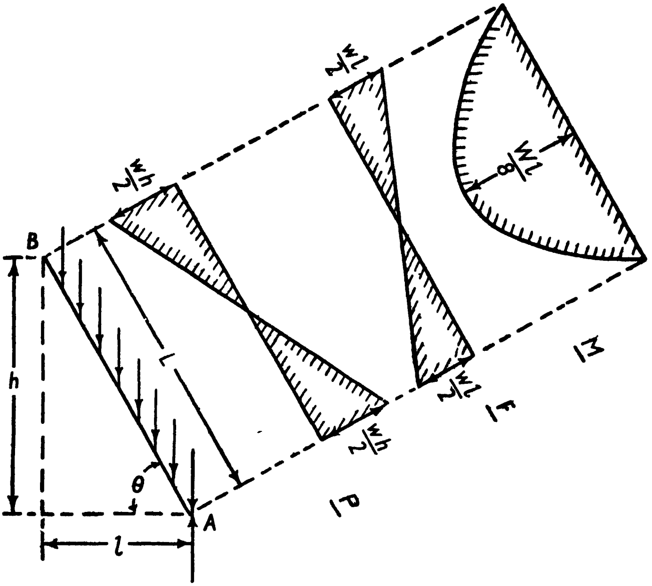


FIG. 64

Resolving the system axially and transversely, the components of the end-reactions are $\frac{wL}{2}\cos\theta$ or $\frac{wl}{2}$ transversely and $\frac{wL}{2}\sin\theta$ or $\frac{wh}{2}$ axially.

The vertical loading w per foot run is equivalent to $w\cos\theta$ or $\frac{wl}{L}$ transversely and $w\sin\theta$ or $\frac{wh}{L}$ axially. The total transverse uniformly distributed load $W_t = \frac{wl}{L} \times L = wl$. The maximum B.M. will be at the middle of the beam, being

$\frac{W_t.L}{8}$ i.e., $\frac{wl.L}{8}$ or $\frac{Wl}{8}$, where W is the weight of the beam and l is its horizontal span.

The diagrams for P, F and M are shown in fig. 64.

Example 5.

A ladder AB 25 feet long, weighing 20 lb. per foot run, rests against a smooth vertical wall with the bottom resting on rough ground. B is 15 feet vertically above A. A man weighing 150 lb. is standing on the ladder at 5 feet from the bottom end. Calculate the reactions at the ground and the wall and construct the axial thrust, shear force and B.M. diagrams for the ladder.

If θ is the inclination of the ladder to the horizontal,

$$\sin\theta = \frac{15}{25} = \frac{3}{5} \text{ and } \cos\theta = \frac{20}{25} = \frac{4}{5}.$$

Let R_1 and R_2 be the normal reactions at the ground and wall respectively and R_f the frictional resistance at the ground. Applying the laws of equilibrium and resolving vertically and horizontally,

$$R_1 = 500 + 150 = 650 \text{ lb-wt.}, \ R_2 = R_f.$$

Taking moments about A,

$$15R_2 = 500 \times 10 + 150 \times 4 = 5600$$

$$R_2 = 373{\cdot}3 \text{ lb-wt.} = R_f.$$

The ladder AB is in equilibrium under a system of vertical and horizontal forces which are all known. Resolve the system axially and transversely to AB.

At B, the axial component

$$P_B = R_2\cos\theta = \frac{373{\cdot}3 \times 4}{5} = 298{\cdot}64 \text{ lb-wt.}$$

And the transverse component

$$F_B = R_2\sin\theta = \frac{373{\cdot}3 \times 3}{5} = 224 \text{ lb-wt.}$$

At A, the axial component,

$P_A = R_1\sin\theta + R_f\cos\theta = 650 \times \frac{3}{5} + 373{\cdot}3 \times \frac{4}{5} = 688{\cdot}64$ lb-wt.

And the transverse component

$F_A = R_1\cos\theta - R_f\sin\theta = 650 \times \frac{4}{5} - 373{\cdot}3 \times \frac{3}{5} = 296$ lb-wt.

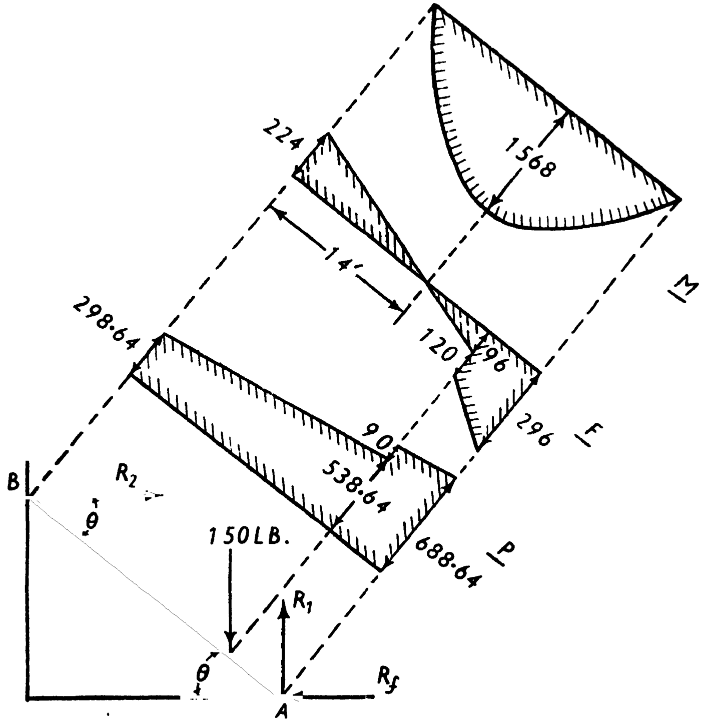


FIG. 65

The rate of loading of 20 lb. per foot length of ladder is equivalent to 20 sinθ or 12 lb. per foot run axially and 20 cosθ or 16 lb. per foot run transversely. Taking the axial and transverse systems of forces separately, the thrust, shear force, and Bending Moment diagrams can now be drawn as shown in fig. 65. The maximum B.M. occurs where the S.F. is zero. From the S.F. diagram it is seen that this section is 14 feet from *B*.

$$M_{\max} = -224 \times 14 + 16 \times 14 \times 7$$
$$= -1568 \text{ lb-feet.}$$

Example 6.

A horizontal beam AB, 12 feet long, is hinged at A and freely supported at B. It carries forces of 500, 800 and 600 lb-wt. at 2, 6 and 9 feet from the hinge and inclined at 45°, 60° and 30° respectively to the horizontal as shown in fig. 66. Construct the normal thrust, shear force and Bending Moment diagrams.

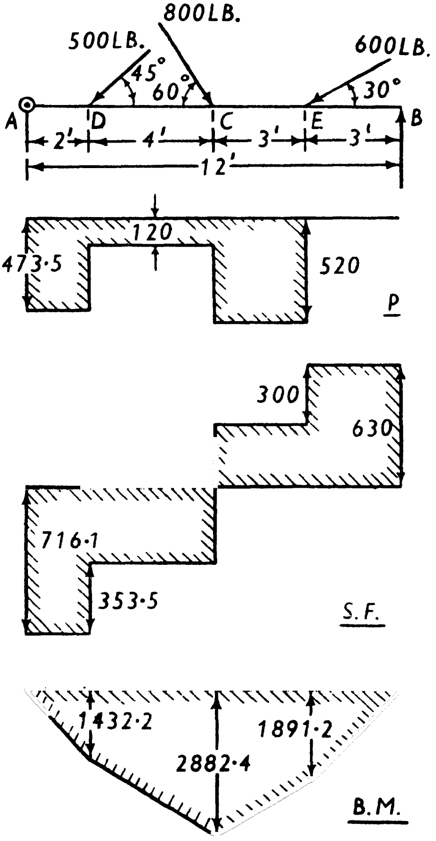


FIG. 66

Resolve the system vertically and horizontally. The vertical system consists of 353·5, 693 and 300 lb-wt. at *D*, *C* and *E* respectively, the vertical reactions at *A* and *B* being, $V_A = 716\cdot1$ lb-wt. and $V_B = 630\cdot4$ lb-wt.

The horizontal system consists of 353·5, 400 and 520 lb-wt. at D, C and E respectively, the force at C being opposite in direction to those at D and E. The horizontal reaction at the hinge is a thrust of 473·5 lb-wt.

The three diagrams can now be drawn.

Example 7.

A vertical post 15 feet high, fixed at the bottom, carries two cranes, one on each side as shown in fig. 67. Construct the normal thrust, shear force and Bending Moment diagrams for the post.

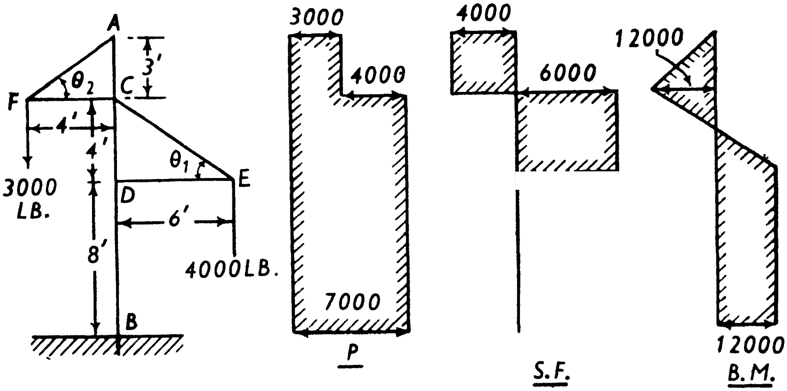


FIG. 67

Let θ_1 and θ_2 be the inclinations of the ties CE and AF to the horizontal.

$\tan\theta_1 = \frac{4}{6} = \frac{2}{3}$ and $\tan\theta_2 = \frac{3}{4}$.

At E, resolving vertically and horizontally,

$ce.\sin\theta_1 = 4000$

$ce.\cos\theta_1 = de$

$\therefore \quad de = 4000 \cot\theta_1 = 4000 \times \frac{3}{2} = 6000$ lb-wt.

The thrust in the member DE is 6000 lb-wt., the pull ce in the tie CE being such that it has to balance a vertical load of 4000 lb. and a horizontal thrust of 6000 lb-wt. at E. Its vertical and horizontal components are therefore 4000 and 6000 lb-wt. respectively.

Similarly at F, resolving vertically and horizontally,

$af \sin\theta_2 = 3000$

$af \cos\theta_2 = cf$

$\therefore \quad cf = 3000 \cot\theta_2 = 3000 \times 4/3 = 4000$ lb-wt.

The thrust *cf* in the member *CF* is 4000 lb-wt., the pull *af* in the tie *AF* being such that it equilibrates the vertical load of 3000 lb. and a horizontal thrust of 4000 lb-wt. at *F*.

The forces on the vertical post *AB* may now be stated as under:

(i) At *A*, the pull *af* which is equivalent to 3000 lb-wt. vertical and 4000 lb-wt. horizontal.

(ii) At *C*, the thrust *cf* of 4000 lb-wt. horizontal accompanied by a pull *ce* from the other side, which is equivalent to 4000 lb-wt. vertical and 6000 lb-wt. horizontal.

(iii) At *D*, the thrust *de* of 6000 lb-wt.

(iv) Reactions at the fixed end *B*.

Starting from *A*, the three diagrams can now be drawn.

8. Fixed Beam with end couples: A beam *AB* of length *l* is fixed at its ends with end-couples M_A and M_B acting on it.

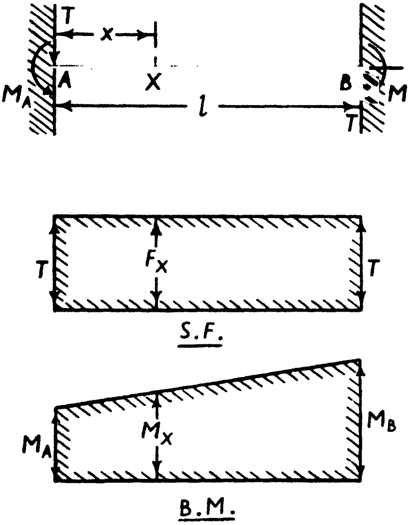


Fig. 68

The unbalanced couple $(M_B - M_A)$, assuming M_B is greater than M_A, can only be balanced by another couple of the same magnitude but opposite in direction.

The reaction at the supports will therefore consist of two equal forces $T - T$, acting upwards at B and downwards at A to offer the balancing couple $T.l$.

$$\therefore \quad T.l = (M_B - M_A)$$

$$\text{and} \quad T = \left(\frac{M_B - M_A}{l}\right)$$

The S.F. diagram is a rectangle as shown in the figure.

Consider a section X distant x from A.

$$M_x = M_A + T.x = M_A + (M_B - M_A) \cdot \frac{x}{l}.$$

The value of M_x at $x = 0$ is M_A and at $x = l$, it is M_B.

The B.M. diagram is thus a straight line whose end ordinates are M_A and M_B.

9. Beam with an intermediate couple: A beam AB of length l is hinged at A and freely supported at B. A couple μ acts on it at C whose distance from A is a and from B it is b, such that $a + b = l$.

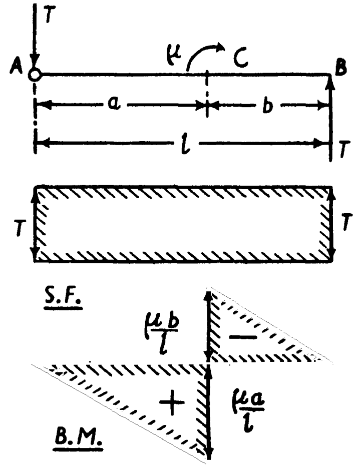


Fig. 69

Construct the S.F. and B.M. diagrams.

The reactions at A and B will be $T - T$ as shown in fig. 69, such that $T.l = \mu$

$$\therefore \quad T = \frac{\mu}{l}.$$

The S.F. diagram is a rectangle having T as its ordinate. Consider a section in the portion AC, distant x from A.

$$M_x = T.x = +\frac{\mu x}{l}$$

At C, $M_{CA} = +\frac{\mu a}{l}$.

For a section between C and B, $M_x = T.x - \mu$.

$$\text{At } C,\ M_{CB} = T.a - \mu = \frac{\mu a}{l} - \mu$$

$$= -\frac{\mu\,(l-a)}{l} = -\frac{\mu b}{l}.$$

The S.F. and B.M. diagrams are shown in fig. 69.

EXAMPLES IV

(1) A cantilever of 5 feet span, carries a load of 2 tons at 2 feet from the fixed end, a load of 1 ton at the free end and a load of 4 tons uniformly spread over the second and third feet of the length, measured from the free end. Construct the S.F. and B.M. diagrams.

($F_{\max} = 7$ tons; $M_{\max} = 21$ ton-feet.)

(2) A cantilever of 6 feet span has a central downward load of 4 tons, an upward force of 1½ tons at the free end and a uniformly distributed load of ½ ton per foot run between the two point-loads. Construct the S.F. and B.M. diagrams indicating the position of the point of contraflexure. (3·56 feet from free end.)

(3) A beam freely supported over a span of 14 feet carries concentrated loads of 2, 3, 4 and 5 tons at 2, 6, 9½ and 11½ feet from the left-hand support. Construct the S.F. and B.M. diagrams.

($M_{\max} = 27{\cdot}77$ ton-feet.)

(4) A beam, 16 feet span, carries a load of 6000 lb-wt. uniformly spread over 4 feet from the centre towards the right. Construct the S.F. and B.M. diagrams. State the position and amount of max. B.M.

(19687·5 lb-feet at 6½ ft. from R.H. support.)

(5) A beam, 15 feet span, carries a uniformly distributed load of ½ ton per foot run over the entire span and two point loads of 4 and 5 tons at 5 and 10 feet from the left hand support. State the position and amount of maximum B.M. and draw the S.F. and B.M. diagrams. (36·7 ton-feet at 6·84 feet from R.H. end.)

(6) A beam, 15 feet long, with a load of ½ ton per foot run over its entire length is hinged at one end and freely supported at 5 feet from the other. It also carries a point-load of 15 tons at the free end. Draw the S.F. and B.M. diagrams, stating the position and amount of max. B.M. (+ 81·25 ton-feet at intermediate support.)

(7) In the above case, the point load of 15 tons is replaced by a point load of 1½ tons at the free end. Construct the S.F. and B.M. diagrams. State the amounts of max. B.M. and position of the point of contraflexure.

(+13·75 ton-feet; — 1·27 ton-feet; 4½ feet from end support.)

(8) A girder, 40 feet long, rests on two supports with equal overhang on both sides and carries a uniformly distributed load of 1½ tons per foot run over the entire length. Calculate the overhang if the maximum B.M. is as small as possible. State the value of this moment and the position of the points of contraflexure.

(8·28 feet from ends; 51·6 ton-feet; 11·7 feet from ends.)

(9) A beam, 50 feet long, rests on supports 40 feet apart with an overhang of 10 feet at the right hand end. The beam weighs ·2 ton per foot run and carries a concentrated load of 15 tons at the middle of its supported length. Calculate the maximum moments and position of the point of contraflexure. Construct the S.F. and B.M. diagrams.

(+10 ton-ft.; — 185 ton-ft.; 10·86 feet from R.H. end.)

(10) A beam, 20 feet long, rests on supports 12 feet apart, the right hand end overhanging its support by 5 feet and the left hand end by 3 feet. The beam carries a uniformly distributed load of 1½ tons per foot run between the supports, a point load of 6 tons at the extreme right hand end and a point load of 4 tons at the left hand end. Construct the S.F. and B.M. diagrams and state the position of the points of inflexion.

$$\left\{\begin{array}{l} M_{\max} = +\ 30 \text{ ton-feet}; -\ 6{\cdot}75 \text{ ton-feet at 8 feet} \\ \text{from L.H. end; 5 and 11 feet from L.H. end.} \end{array}\right\}$$

(11) A girder 30 feet long rests on supports 20 feet apart, with an overhang of 6 feet on left and 4 feet on right. It carries a uniform load of 2 tons per foot run from the left hand end to the right hand support and a concentrated load of 30 tons at the right hand end. Construct the S.F. and B.M. diagrams and state the positions of the points of inflexion.

$$\left\{\begin{array}{l} M_{\max} = +\ 120 \text{ ton-ft.}; -26{\cdot}6 \text{ ton-ft at 13·9 feet} \\ \text{from L.H. end; 8·78 and 19·02 feet from L.H. end.} \end{array}\right\}$$

(12) A girder 40 feet long is supported over 25 feet with an overhang of 10 feet on the left and 5 feet on the right. The girder weighs ¼ ton per foot run and has a load of 6 tons at the L.H. end, 10 tons at the R.H. end and a concentrated load of 30 tons at 10 feet from the L.H. support. Draw the S.F. and B.M. diagrams.

($M_{\max} = +\ 72{\cdot}5$ ton-ft.; — 134 ton-feet at the 30-ton load.)

(13) A timber beam 25 feet long and 15″ square in section floats horizontally in sea water. The weight of timber is 40 lb. per cubic foot and of water 64 lb. Two equal weights just sufficient to immerse the beam are placed on it 7 feet from each end. Calculate the value of each weight. Draw the S.F. and B.M. diagrams for the beam and state the value of maximum B.M.

(468·75 lb-wt.; 918·75 lb-feet.)

(14) In the above case, if the two point loads are replaced and instead a point load of 500 lb-wt. is placed at 5 feet from the L.H. end, what should be the value of another point load and where should it be placed so that the beam is just immersed and retains its horizontal position? Draw the S.F. and B.M. diagrams.

(437·5 lb-wt. at 3·93 feet from R.H. end.)

(15) A beam 20 feet long is freely supported at the ends and carries a uniformly distributed load of ½ ton per foot run throughout the span and an additional load of 1½ tons per foot run for 14 feet from the L.H. end. Find the position and magnitude of maximum B.M. and draw the S.F. and B.M. diagrams.

(86·92 ton-feet at 9·33 feet from L.H. end.)

(16) An 18″ thick and 8′ high wall, triangular in elevation with one end vertical rests on whole of a 20′ span beam which is freely supported at each end. Calculate the position and amount of max. B.M., if masonry weighs 120 lb. per cub. foot. Calculate the S.F. and B.M. at 2 feet intervals and plot the diagrams.

(37000 lb-ft. at 11·55 feet from L.H. end.)

(17) A wall triangular in elevation, covers a girder, the apex of the wall being over the middle of the span. If the span is 20 feet and the central height of wall is 12 feet, its thickness being 18″, calculate the S.F. and B.M. for sections at 2 feet intervals and plot the diagrams. Density of masonry = 130 lb. per cubic foot.

(M_{max} = 78000 lb-feet at centre.)

(18) A single skin cofferdam is subjected to a head of 18 feet of water. If the plate is held in a vertical position by two horizontal supports one at the surface of water at the top and the other 18 feet below at the bottom, find the position and amount of maximum B.M. on one-foot width of sheeting. Weight of water = 64 lb. per cubic foot. Calculate the S.F. and B.M. at 2 feet intervals and plot the diagrams.

(24000 lb-feet at 10·39 feet from top.)

(19) A steel plate bulkhead 15 feet high is stiffened by vertical beams, spaced 30″ centre to centre. They are supported horizontally at top and bottom. If the compartment on one side of the bulkhead is filled with water, calculate the position and amount of maximum B.M.

for one of the stiffeners, if the density of water is 62·5 lb. per cubic foot. Calculate the S.F. and B.M. at 3 feet intervals and plot the diagrams.
($M_{\max} = 33820$ lb-feet at 8·66 feet from top.)

(20) A horizontal beam, 16 feet long, freely supported at the ends, carries a concrete wall 9″ thick; its height is 6′ at one end and increases uniformly to 10′ at the other. If density of concrete is 140 lb. per cubic foot, find the position and amount of maximum B.M. Construct the S.F. and B.M. diagrams for the beam.
(26930 lb-feet at 8·32′ from L.H. end.)

(21) The beam *AB*, shown in fig. 70, is 12 feet long and is hinged to a wall at *A* and supported horizontally by a strut *DE*, 10 feet long, hinged to the wall at *E*, 10 feet vertically below *A*, and at *D* to a vertical projection *CD*, 2 feet long, which is rigidly attached to the beam at its midpoint *C*. The beam carries a uniformly distributed load of one ton per foot run over the whole length and a point load of 5 tons at the free end *B*. Calculate the reactions at the hinges *A* and *E*, the thrust in the strut *DE*, and draw the S.F. and B.M. diagrams for the beam *AB*.

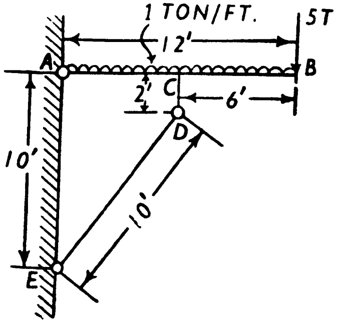


FIG. 70

$$\left\{\begin{array}{l} V_A = \cdot 6 \text{ ton downwards; } V_E = 17\cdot60 \text{ tons upwards; } H = 13\cdot2 \text{ tons;} \\ de = 22 \text{ tons; } M_{CB} = +48 \text{ ton-feet; } M_{CA} = +21\cdot6 \text{ ton-ft.} \end{array}\right\}$$

(22) A beam, 24 feet span, is subjected to end moments of 30 and 50 ton-feet. The couples act at the ends in opposite directions. Construct the S.F. and B.M. diagrams.
($T = \frac{5}{6}$ ton-wt.)

(23) A beam *AB*, 20 feet long, is hinged at *A* and freely supported at *B*. A clockwise couple of 15000 lb-feet is applied at *C* at a distance of 15 feet from the hinge *A*. Calculate the reactions at the support and construct the S.F. and B.M. diagrams.
($T = 750$ lb-wt.)

(24) The cantilever in question 1 has in addition, a clockwise couple of 10 ton-feet applied to it at the free end. Construct the B.M. diagram.

($M_{\max} = 31$ ton-feet.)

(25) The beam in question 23, also carries a uniformly distributed load of 800 lb. per foot run. Calculate the end reactions and draw the S.F. and B.M. diagrams.

$$\left\{\begin{array}{l} 7250 \text{ lb-wt. at L.H.; } 8750 \text{ lb-wt. at R.H. end;} \\ M_{CB} = -33750 \text{ lb-ft.; } M_{CA} = -18750 \text{ lb-feet;} \\ M_{\max} = -32850 \text{ lb-feet where S.F. is zero.} \end{array}\right\}$$

CHAPTER V

BEAMS AND BENDING — II.

Stresses in Beams

1. **Stresses in a beam section:** In the preceding chapter we have studied the effects of a load system on the cross-section of a beam, viz., the Shear Force and the Bending Moment. These have to be resisted and, as usual, the structural member — in this case the beam — has to deform itself. As it bends, resistances to the actions are set up and the process of bending will stop when every cross-section has set up full resistance to the Shear Force and Bending Moment acting on it. At any point in the cross-section of a beam, there will thus be (i) a stress to resist the Bending Moment and (ii) a stress to resist the Shear Force. We shall first take up the resistance to the Bending Moment, as it is the more important of the two.

2. **Theory of Simple Bending:** Consider a strip δx of the beam subjected to a bending moment M. All the layers of the strip from top to bottom will bend. In figure 71(a), the top layer AC has bent to A_1C_1, GH to G_1H_1 and the bottom layer BD to B_1D_1. All these layers which were originally of the same length δx are no longer so. The top one has been compressed, its length now being A_1C_1. As we proceed lower, the layers have suffered compression but to a lesser degree until we come across the layer EF which has suffered no change in its length, though it has got bent into E_1F_1. Below EF the layers are stretched, the amount of extension increasing as we proceed lower, until finally, we get to the bottom-most layer BD which has stretched to B_1D_1. The layers above EF have been compressed, those below EF have stretched, the amount of compression or extension depending upon the position of the layer with regard to EF which is known as the neutral layer because it has neither stretched nor has it been compressed.

Let us consider the deformation of some particular layer, say GH, which is at a distance y from the neutral layer EF. The strain of this layer is,

$$e = \frac{GH - G_1H_1}{GH}, \text{ compressive.}$$

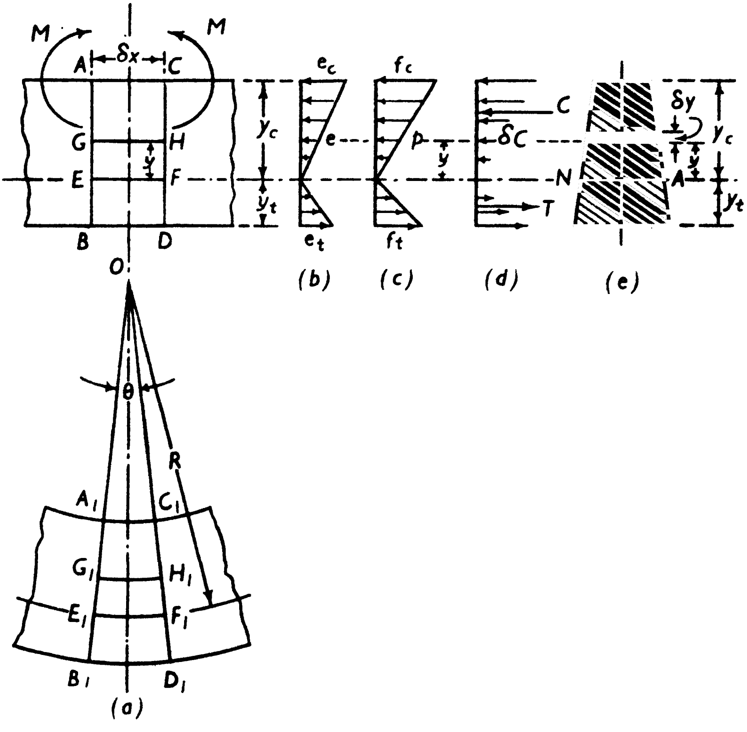


Fig. 71

The strip δx has bent and the various layers from AC down to BD have taken up their respective shapes, the nature of curvature depending upon the moment M. Any infinitesimal length of a curve is equivalent to and therefore can be looked upon, as an arc of a circle of a definite radius. A_1C_1, G_1H_1, E_1F_1 and B_1D_1 are not necessarily circular arcs but if δx is very small, they can be looked upon as arcs of concentric circles with centre O, the intersection of B_1A_1 and D_1C_1 produced. Thus E_1F_1 will be treated as an arc of a circle with centre O and radius OE_1. O is called the centre of curvature and OE_1 the radius of curvature.

Let $OE_1 = R$. Then, arc $E_1F_1 = R.\theta$, where θ is the angle subtended by the arc at O. But since E_1F_1 has suffered no change in length, $E_1F_1 = R.\theta = \delta x$; $G_1H_1 = OG_1.\theta = (R-y).\theta$, since the distance of the layer from $E_1F_1 = y$. The original length GH of the layer is δx which is $R.\theta$.

The strain e of the layer is therefore given by,

$$e = \frac{GH - G_1H_1}{GH} = \frac{R.\theta - (R-y)\theta}{R.\theta} = \frac{y}{R}$$

We therefore deduce that the strain of any layer is proportional to its distance from the neutral layer, since R, the radius of curvature of the neutral layer at a given section is constant. The strain is compressive or tensile according as the layer is above or below the neutral layer, the maximum strain being at the top and bottom layers. If y_c and y_t are respectively their distances from the neutral layer, the corresponding strains e_c and e_t are given by,

$$e_c = \frac{y_c}{R} \text{ and } e_t = \frac{y_t}{R}$$

The variation of strain, which is a straight line is shown in fig. 71(*b*).

3. Neutral Axis: The intersection of the plane of the neutral layer with the cross-section of the beam is called the Neutral Axis. The intensity of stress at a point in the cross-section at a distance y from the neutral axis can now be deduced. Within the Elastic limit, stress is proportional to strain. The intensity of stress p, therefore, is equal to $E.e$ or $\frac{E.y}{R}$ where E is the modulus of Elasticity, p being compressive if the point is above the neutral axis and tensile if below it. Obviously, since the strain at any point is proportional to its distance from the neutral axis, the corresponding stress is also proportional to the distance from the neutral axis, E and R being constant.

The distribution of stress is shown in fig. 71 (*c*) alongside the strain variation. If f_c and f_t are the stresses at the top and bottom of the section,

$$f_c = \frac{E.y_c}{R} \text{ and } f_t = \frac{E.y_t}{R}, \text{ since } \frac{p}{y} = \frac{E}{R}$$

These stresses which are set up are intended to resist the bending moment M at the cross-section. Now, a couple can only be balanced by a couple. So the resistances set up by the section must reduce to a couple which will be numerically equal to M but opposite in direction.

Consider an elementary area δa of the section at a height y above the N.A. [fig. 71 (*e*)]. The intensity of stress p on the area is sensibly $\frac{E.y}{R}$ if its thickness δy is small. The thrust δC exerted on the area is equal to $p.\delta a$. If we add up all the thrusts on the elementary areas above the N.A., they will reduce to a single resultant thrust C. Similarly, all the elementary pulls $\delta T = p.\delta a$ on areas below the N.A. will add up into a single resultant pull T. C and T will be equal and will constitute a couple to resist the bending moment M [fig. 71 (*d*)].

If we sum up all the elementary resistances $p.\delta a$ from top to bottom, the result must be zero, since $C = T$.

Thus, $$\sum_{y_t}^{y_c} p.\delta a = 0$$

∴ $$\sum_{y_t}^{y_c} \frac{E}{R}.y.\delta a = 0$$

or $$\sum_{y_t}^{y_c} y.\delta a = 0.$$

The expression $y.\delta a$ stands for the moment of the elementary area δa about the N.A. and $\Sigma y.\delta a$ represents the sum of the moments of all the elementary areas about the N.A. But this must equal the moment of the sectional area A about the N.A. We therefore get,

$$\sum_{y_t}^{y_c} y.\delta a = A.\bar{y} = 0,$$

where $\bar{y}$ is the distance of the centre of gravity of the area of cross-section from the N.A.

$\therefore$ $\bar{y} = 0$, i.e. the centre of gravity of the sectional area lies on the N.A.

In the preceding reasoning, we had postulated a neutral layer and therefore the neutral axis, but about its exact location, we were not in a position to make any statement. Now we know that the neutral axis passes through the centre of gravity of the area of cross-section.

4. Moment of Resistance: To obtain the Moment of resistance, one way would be to work out the value of C or T and calculated the lever arm of the couple. Then, the moment of resistance $M_r = C$ or $T \times a$. There is another and easier way to calculate the moment of resistance.

The resistances set up by elementary areas δa of the cross-section are all parallel forces, thrusts above the N.A. and pulls below it. The algebraic sum of the moments of all these forces about any line must be equal to the moment of their resultant about the same line. The resultant in this case, is a couple and the moment of a couple about any line is the couple itself. Take moments of all the elementary pulls and pushes about the neutral axis. Their algebraic sum will necessarily be equal to the resultant couple or the moment of resistance which we desire to evaluate. Thus,

$$M_r = \sum_{y_t}^{y_c} p.\delta a.y = \sum_{y_t}^{y_c} \frac{E}{R}.y\delta a.y$$

$$= \frac{E}{R}\sum_{y_t}^{y_c} y^2.\delta a.$$

If each elementary area δa is multiplied by the square of its distance from the N.A., the summation of these products is $\Sigma y^2.\delta a$. This is purely a mathematical expression and is called the Moment of Inertia or the Second Moment of the section about the N.A. and is usually denoted by the letter I. The expression for the Moment of Resistance, therefore becomes,

$$M_r = \frac{EI}{R}$$

$$\therefore \quad \frac{M_r}{I} = \frac{E}{R}.$$

We have already shown that $\frac{p}{y} = \frac{E}{R}$.

We may combine the results and state that $\frac{M_r}{I} = \frac{p}{y} = \frac{E}{R}$. This result is very important and should be remembered. The relation $M_r = \frac{p.I}{y}$ expresses the moment of resistance in terms of the stress at any point in the cross-section and the dimensions of the cross-section. For purposes of design, it is desirable to express the moment in terms of maximum stresses which occur at the top or bottom of the section.

Thus, $M_r = \frac{f_c}{y_c} \times I = \frac{f_t}{y_t} \times I$.

The expression can be further simplified into

$M_r = f_c Z_c = f_t Z_t$, Z_c being $\frac{I}{y_c}$ and Z_t being $\frac{I}{y_t}$.

The term Z which stands for the moment of inertia of the section about the N.A. divided by the extreme value of y is called the "Modulus of section".

If the section of the beam is perfectly symmetrical, its centre of gravity and therefore the N.A. will be at the middle of its depth d. In this case,

$$y_c = y_t = \frac{d}{2}$$

and $$Z_c = Z_t = \frac{I}{d/2}$$

$$\therefore \quad f_c = f_t.$$

If we denote the extreme intensity of tensile or compressive stress by f and the section-modulus $\frac{I}{d/2}$ by Z, the expression for the moment of resistance is $M = f.Z$.

5. Assumptions made in the simple Theory of Bending: We have worked out the key formula $\frac{M_r}{I} = \frac{p}{y} = \frac{E}{R}$, in this theory of simple Bending. We may now review the reasoning and take note of the assumptions tacitly made in evolving the formula.

In figure 71, the plane sections *AB* and *CD* before bending have been shown after bending, as plane sections A_1B_1 and C_1D_1. Thus, to begin with, an assumption has been made that transverse sections which were plane before bending remain plane after bending. We worked out the strain e of a layer *GH* and deduced that the stress $p = E.e.$ This is true only if the Elastic Limit for the material is not exceeded. Further, we have assumed that each layer is free to expand or contract independently of the layers above and below it. Otherwise, E in the expression $p = E.e$ will not be the pure modulus of Elasticity but some modified form. In summing up $\Sigma p.\delta a$ from top to bottom we substituted $\frac{E}{R}.y$ for p. We tacitly assumed that the modulus of Elasticity in tension and compression was the same.

We may summarise the assumptions as under:

(i) Transverse sections which are plane before bending remain plane after bending.

(ii) The material is homogeneous and isotropic, i.e., it possesses the same elastic properties in all directions and the Elastic limit is not exceeded.

(iii) Each layer is free to expand or contract independently of the layer above or below it.

(iv) The value of E is the same in tension and compression.

For this theory of simple bending, the beam section must be symmetrical about the plane of bending.

Example 1.

Find the maximum intensity of stress induced in a steel flat 6" wide × ½" thick if it is bent into a circular arc of 40 feet radius. $E = 13000$ *tons/in²*.

$$\frac{M}{I} = \frac{p}{y} = \frac{E}{R}.$$

At the inner and outer surface, the value of y is $\frac{t}{2}$ or $\frac{1}{4}''$.

$$\therefore \quad f = \frac{E}{R} \times \frac{t}{2} = \frac{13000}{480} \times \frac{1}{4} = 6{\cdot}77 \text{ tons/in}^2.$$

Example 2.

A beam of symmetrical section is 14″ deep and has a moment of Inertia of 442·57 inch units about its principal axis. To what radius may it be bent if the maximum skin-stress is not to exceed 8 tons/in^2? Take E = 13000 tons/in^2. What would be the moment of resistance at this stress?

$$\frac{M_r}{I} = \frac{f}{d/2} = \frac{E}{R}$$

$$\therefore \quad R = \frac{E.d}{2f} = \frac{13000 \times 14}{2 \times 8} = 11375 \text{ inches or } 947{\cdot}9 \text{ feet.}$$

Also $M_r = f \times \frac{I}{d/2} = 8 \times \frac{442{\cdot}57}{7} = 505{\cdot}8$ ton-inches.

Example 3.

A beam of symmetrical section, depth = 16″, I = 618 inch units, is simply supported over a span of 24 feet. What uniformly distributed load may it carry if the maximum bending stress is not to exceed 7½ tons/in^2.? What concentrated load may be carried by the beam at the centre with the same permissible skin-stress?

$$Z = \frac{I}{d/2} = \frac{618}{8} = 77{\cdot}25 \text{ inch units.}$$

At 7½ tons/in^2. bending stress, the moment of resistance of the beam $M_r = f.Z = 7{\cdot}5 \times 77{\cdot}25 = 579{\cdot}375$ ton-inches.

If a beam carries a total load W uniformly spread over its span l, the maximum B.M. is $\frac{Wl}{8}$ at the central section.

If W is the load in tons, the maximum B.M. is $\frac{W\times 24}{8}$ or $3W$ ton-feet i.e. $36W$ ton-inches.

Since the moment of resistance has to balance the bending moment, the two are numerically equal.

$$\text{Equating, } 36W = 579{\cdot}375$$

$$\therefore \quad W = 16{\cdot}1 \text{ tons}$$

$$\text{or} \quad {\cdot}671 \text{ ton per foot run.}$$

If the load W is concentrated at the centre, the maximum B.M. is $\frac{Wl}{4}$ at the central section,

i.e. $\frac{24W}{4} \times 12$ or $72W$ ton-inches.

Equating the bending moment to the moment of resistance,

$$72W = 579{\cdot}375$$

$$\therefore \quad W = 8{\cdot}05 \text{ tons.}$$

Example 4.

A beam of symmetrical section has a depth of 20″ and a moment of inertia of 1673 inch units about its axis of bending. Find the maximum permissible span for this beam if, simply supported at the ends, it has to carry a uniformly distributed load of ¾ ton per foot run without exceeding a bending stress of 8 tons/in^2.

The section-modulus $Z = \frac{I}{d/2} = \frac{1673}{10} = 167{\cdot}3$ inch-units.

The moment of resistance offered by the section at a stress of 8 tons/in^2. $= f.Z$ i.e. $8 \times 167{\cdot}3$ or $1338{\cdot}4$ ton-inches.

If l is the permissible span in feet, the total load $W = wl$ or $\frac{3l}{4}$ tons.

The maximum B.M. at centre $= \frac{Wl}{8}$

i.e. $\frac{3l^2}{32}$ ton-feet or $\frac{9l^2}{8}$ ton-inches.

Equating the bending moment to the moment of resistance

$$\frac{9l^2}{8} = 1338{\cdot}4$$

$$\therefore \quad l^2 = 1189{\cdot}68$$

$$\text{or} \quad l = 34{\cdot}49 \text{ feet.}$$

Important:

Note 1. The stress is specified in terms of a unit area, usually one square inch. The dimensions of a beam section are also usually in inch units. The modulus of section is thus in inch units. The expression for the moment of resistance $f.Z$ will also be in inch-units. On the other hand, the bending moment, depending as it does on the span, is usually in feet units as the span is specified in these units. In equating the bending moment to the moment of resistance, care must be taken to convert the bending moment to inch-units.

Note 2. The expression for the moment of Inertia is $\Sigma y^2.\delta a$. If the dimensions of the section are in inch-units, the units of moment of Inertia will be in^2. × in^2. or in^4. The units of the modulus of section which is $\frac{I}{d/2}$ will be $\frac{\text{in}^4.}{\text{in.}}$ i.e. in^3.

6. Beams of rectangular section: In the examples worked out above, the moments of Inertia of beam sections were supplied but we shall be able to use the formula $M_r = f.Z$ freely when we know how to calculate for ourselves the moments of Inertia and moduli of section of the common beam sections. The rectangular section which is common for timber joists, is so simple that it is possible to arrive at its modulus of section without recourse to its moment of Inertia about the axis of bending.

Let b be the width and d the depth of the section. Divide the sectional area into a number of thin strips parallel to the N.A., of the same thickness δy. Consider an elementary strip at a height y above the N.A. The area of the strip $\delta a = b.\ \delta y =$ constant, since all the strips have the same width and thickness. The stress-intensity p on this area in terms of the extreme stress f is given by

$$\frac{p}{y} = \frac{f}{d/2} \quad \text{or} \quad p = \frac{2f.y}{d}.$$

The elementary thrust on the area δa is given by

$$\delta C = p.\delta a = \frac{2fy}{d}.\delta a.$$

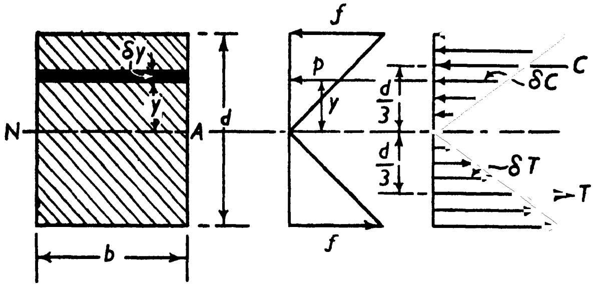


Fig. 72

Since f, d, δa are all constant, δC varies as y. If we draw a diagram of thrusts for all areas from $y = 0$ to $y = \frac{d}{2}$, it will be a straight line as shown in fig. 72.

The resultant of all these parallel thrusts will be C = average intensity of stress × area under compression $= \frac{f}{2} \times \frac{b.d}{2}$ $= \frac{f.b.d}{4}$, acting at the centre of gravity of the triangle of thrusts i.e. at $\frac{2}{3} \times \frac{d}{2}$ or $\frac{d}{3}$ from the N.A.

Similarly, all the parallel pulls below the N.A will add up to a single resultant force $T = \frac{f}{2} \times \frac{b.d}{2} = \frac{fbd}{4}$ acting at $\frac{d}{3}$ from the N.A.; C and T constitute the couple of resistance whose arm a is $\frac{d}{3} + \frac{d}{3}$ i.e. $\frac{2d}{3}$.

Its moment is C or $T \times a = \frac{fbd}{4} \times \frac{2d}{3} = f \times \frac{1}{6}\, bd^2$

The section-modulus for a rectangular section is therefore $\frac{1}{6}\, bd^2$.

Example 5.

A timber joist of rectangular section 6" wide × 12" deep is freely supported over a span of 16 feet and carries a uniformly distributed load of 6000 lb-wt. Calculate the skin stresses at 2, 4, 6 and 8 feet from the left-hand support.

The modulus of section $= \frac{1}{6} \times 6 \times 12 \times 12 = 144$ in^3. The bending moments at the specified sections will be:

At 2 feet, $M_2 = 3000 \times 2 - \frac{6000}{16} \times 2 \times 1 = 5250$ lb-feet.

$= 63000$ lb-inches.

At 4 feet, $M_4 = 3000 \times 4 - \frac{6000}{16} \times 4 \times 2 = 9000$ lb-feet.

$= 108000$ lb-inches.

At 6 feet, $M_6 = 3000 \times 6 - \frac{6000}{16} \times 6 \times 3 = 11250$ lb-feet.

$= 135000$ lb-inches.

At 8 feet, $M_8 = 3000 \times 8 - \frac{6000}{16} \times 8 \times 4 = 12000$ lb-feet.

$= 144000$ lb-inches.

Since at any section, $M_r = f.Z$, the extreme fibre stresses at the specified sections will be:

$$f_2 = \frac{M_2}{Z} = \frac{63000}{144} = 437{\cdot}5 \text{ lb/in}^2.$$

$$f_4 = \frac{M_4}{Z} = \frac{108000}{144} = 750 \text{ lb/in}^2.$$

$$f_6 = \frac{M_6}{Z} = \frac{135000}{144} = 937{\cdot}5 \text{ lb/in}^2.$$

$$f_8 = \frac{M_8}{Z} = \frac{144000}{144} = 1000 \text{ lb/in}^2.$$

The stress distribution across the four sections is shown in fig. 73.

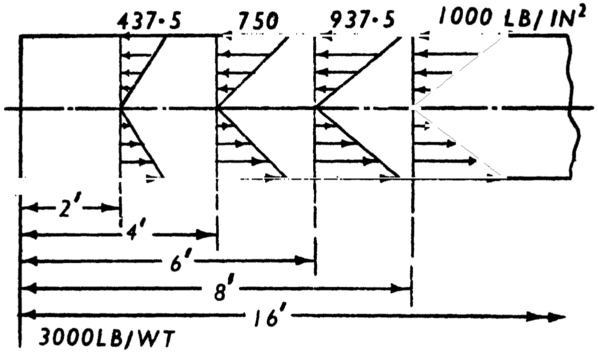


FIG. 73

This is a timber beam for which the usual permissible stress is 1000 lb/in^2. or $\frac{1}{2}$ ton/in^2. The beam section is therefore quite adequate to take the load of 6000 lb. spread over a span of 16 feet, the maximum fibre-stress induced being just 1000 lb/in^2. at the central section. At other sections, the bending moment is less and since the same section has been provided i.e. the modulus of section Z is the same at all sections, the skin stress will be correspondingly less. It might then appear that material could have been saved and economy effected if the section had been reduced to suit the bending moment so that the skin stress was the maximum permissible for the material. A little thought however, will show that for ordinary spans and loads, it is much better to have beams of the same section throughout because a little saving in material is more than counterbalanced by the labour and cost involved in cutting a beam of a carefully designed variable section. For long spans and heavy loads, however, as in the case of bridge-girders, it is wasteful to provide the same section throughout and it is varied to suit the Bending Moment it has to resist. Such beams of varying section are known as beams of uniform strength and will be discussed later.

7. Moments of Inertia of sections: We shall now proceed to work out moments of Inertia of common sections.

To begin with, we shall take note of two very useful propositions.

I. Moments of Inertia about parallel axes.

If the moment of Inertia of a plane area about an axis *GG* (in its plane) passing through the centre of gravity of area is denoted by I_{GG}, the moment of Inertia of the area about a parallel axis *AB*, at a distance h from *GG* is given by $I_{AB} = I_{GG} + A.h^2$, where A is the area of section. This is the well-known parallel-axes theorem and is easily proved.

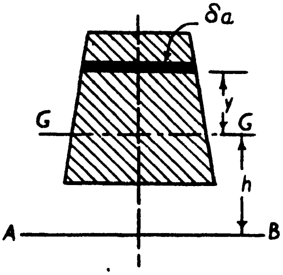


Fig. 74

Consider an elementary area δa at a distance y from *GG*. Then, $I_{GG} = \Sigma y^2.\delta a$. Also $I_{AB} = \Sigma(h \pm y)^2.\delta a$, the minus sign being taken if the area δa is below *GG*.

$$\therefore \quad I_{AB} = \Sigma(h^2 + y^2 \pm 2hy)\ \delta a$$
$$= \Sigma h^2.\delta a + \Sigma y^2.\delta a \pm 2h\Sigma y.\delta a$$
$$= A.h^2 + I_{GG} \pm 0.$$

$\Sigma y.\ \delta a$ is the algebraic sum of the moments of all elementary areas δa about the axis *GG*. This is equal to $A.\bar{y}$ where $\bar{y}$ is the distance of the centre of gravity of A from *GG*. But as *GG* passes through the centre of gravity of the area, $\bar{y}$ must be zero. The proposition is therefore proved.

II. If I_{xx} and I_{yy} are the moments of Inertia of a plane section about two axes *XX* and *YY* in its plane at right angles to each other, the moment of Inertia I_{zz} of the section about the axis *ZZ* perpendicular to the plane and passing through *O* — the intersection of *XX* and *YY* — is given by $I_{zz} = I_{xx} + I_{yy}$.

Consider an elementary area δa of the section, its coordinates referred to XX and YY as axes of reference, being (x,y).

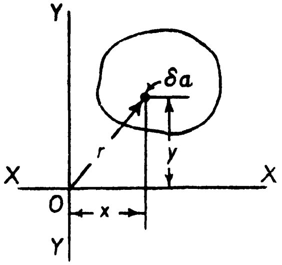


FIG. 75

The distance of the area from the origin O or the axis ZZ through O is given by,

$$r^2 = x^2 + y^2.$$

Now, $I_{xx} = \Sigma y^2.\delta a$ and $I_{yy} = \Sigma x^2 \delta a$.

$$\therefore \quad I_{zz} = \Sigma r^2.\delta a = \Sigma\ (y^2 + x^2)\ \delta a = \Sigma y^2 \delta a + \Sigma x^2.\delta a$$
$$= I_{xx} + I_{yy}.$$

(i) Rectangular section: breadth b, depth d.

Divide the section into a series of strips parallel to XX. Consider a strip of thickness δy at a distance y from XX.

$$\delta a = b.\delta y$$

$$I_{xx} = \Sigma y^2.\delta a = \int_{-d/2}^{+d/2} y^2.b.dy$$
$$= 2b \int_0^{d/2} y^2.dy = 2b \left\{\frac{y^3}{3}\right\}_0^{d/2}$$
$$= \frac{bd^3}{12}.$$

The modulus of section $Z_{xx} = \dfrac{I_{xx}}{d/2} = \dfrac{1}{6}\ bd^2$.

If the moment of Inertia of the section about a side, say AB, is required, it can be easily derived by the parallel-axes theorem.

Thus,

$$I_{AB} = I_{xx} + A.h^2 = \frac{1}{12}bd^3 + bd\left(\frac{d}{2}\right)^2$$

$$= \frac{1}{12}bd^3 + \frac{1}{4}bd^3 = \frac{1}{3}bd^3.$$

The moment of Inertia of the section about the axis YY will be,

$$I_{yy} = \frac{1}{12}db^3.$$

It is best to remember that the moment of Inertia of a rectangular section about a principal axis is $\frac{1}{12}$ × width × (depth)^3. (|| to principal axis width)

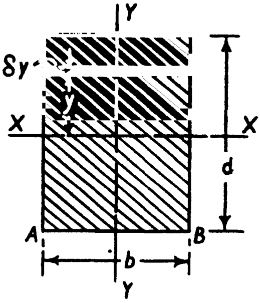


FIG. 76

If the moment of Inertia of an area is expressed as $I = A.k^2$, the term k is called the radius of gyration of the area about the axis. This is very useful because if the "k" of a section is given, we obtain I immediately by working out Ak^2.

For the rectangular section,

$$I_{xx} = \frac{1}{12}bd^3 = A.k_{xx}^2 = bd.k_{xx}^2$$

$$\therefore \quad k_{xx}^2 = \frac{1}{12}d^2 \text{ or } k_{xx} = \frac{d}{2\sqrt{3}}$$

Similarly,

$$k_{yy}^2 = \frac{1}{12}b^2 \text{ or } k_{yy} = \frac{b}{2\sqrt{3}}$$

(ii) Hollow rectangular section; *I*-section; Channel section.

Let the dimensions of the hollow rectangular section be $B \times D$ externally, the internal width and depth being b and d respectively.

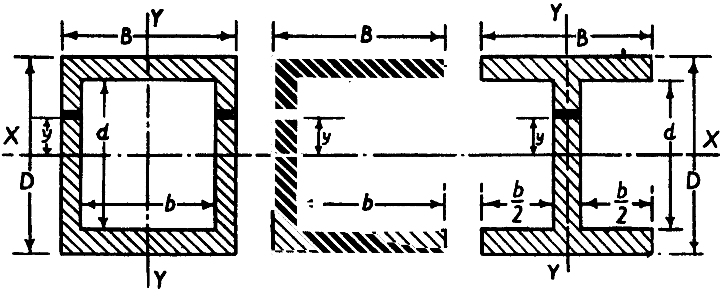


FIG. 77

The moment of Inertia of a hollow rectangular section about the axis XX is the difference of the moments of Inertia of two solid rectangular sections of dimensions $B \times D$ and $b \times d$.

$$\therefore \quad I_{xx} = \frac{1}{12}B.D^3 - \frac{1}{12}b.d^3.$$

The modulus of section $$Z_{xx} = \frac{I_{xx}}{y_{max}} = \frac{BD^3 - bd^3}{12} \div \frac{D}{2}$$

$$= \frac{BD^3 - bd^3}{6D}$$

Also $$I_{xx} = A.k_{xx}^2$$

$$\therefore \quad k_{xx}^2 = \frac{BD^3 - bd^3}{12\ (BD - bd)}.$$

If the area of a hollow rectangular section is re-arranged either by (i) shifting a side-web to the other end or by (ii) shifting both side-webs to the centre, we obtain the channel section and the *I*-section respectively.

The re-arrangement does not affect the value of any elementary area δa of the web nor its distance y from the axis XX. And since $I_{xx} = \Sigma y^2.\delta a$, the moment of Inertia I_{xx} of a channel section or an *I*-section about the axis XX is $\dfrac{(BD^3 - bd^3)}{12}$.

In the case of the *I*-section, b is the difference between the over-all width and the thickness of the web.

The moment of Inertia of the hollow rectangular section about the axis YY is given by, $I_{yy} = \dfrac{DB^3 - db^3}{12}$, but that of the channel or *I*-section will not be the same, since in shifting the side webs, though δa remains the same, its distance from the axis YY is altered.

(iii) Circular section.

(a) Solid circular section: diameter d.

Split up the section into a number of strips parallel to XX and consider an elementary strip of thickness δy at a distance y from XX such that $y = r\sin\theta$, r being the radius of section.

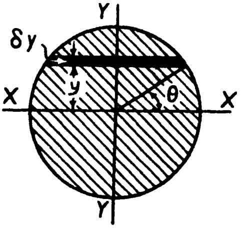


FIG. 78

The area of strip, $\delta a = 2r.\cos\theta.\delta y$.

Since $\delta y = r.\cos\theta.\delta\theta$,

$\delta a = 2r^2.\cos^2\theta.\delta\theta$

$$I_{xx} = \Sigma y^2.\delta a = \int_{-\pi/2}^{+\pi/2} r^2\sin^2\theta.2r^2\cos^2\theta.d\theta.$$

$$= 4r^4\int_0^{\frac{\pi}{2}}\sin^2\theta\ (1-\sin^2\theta)\ d\theta = 4r^4\left\{\int_0^{\frac{\pi}{2}}\sin^2\theta d\theta - \int_0^{\frac{\pi}{2}}\sin^4\theta d\theta\right\}$$

$$= 4r^4\left\{\frac{1}{2}.\frac{\pi}{2}-\frac{3}{4}.\frac{1}{2}.\frac{\pi}{2}\right\} = \frac{\pi r^4}{4} = \frac{\pi d^4}{64}$$

$$Z_{xx} = \frac{I_{xx}}{d/2} = \frac{\pi}{32}d^3.$$

Also, $I_{xx} = A.k_{xx}^2$

$$\therefore \quad k_{xx}^2 = \frac{d^2}{16} \text{ or } k_{xx} = \frac{d}{4} \text{ i.e. } \frac{r}{2}.$$

The moment of Inertia of the circular section is derived more easily by dividing it up into a series of elementary concentric rings and calculating the moment of inertia about the axis ZZ.

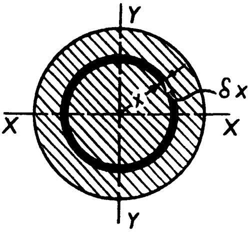


FIG. 79

Consider an elementary ring of radius x and thickness δx. The area of the ring $\delta a = 2\pi x.\delta x$. The moment of inertia of a thin ring or hoop about its central axis is area × radius². The moment of Inertia of the section about the central axis is therefore given by,

$$I_{zz} = \Sigma x^2.\ \delta a = \sum_{0}^{r} x^2.2\pi x\delta x = 2\pi\int_{0}^{r} x^3 dx$$

$$= 2\pi.\frac{r^4}{4} = \frac{\pi r^4}{2} \text{ or } \frac{\pi d^4}{32}.$$

But, $I_{zz} = I_{xx} + I_{yy}$

$= 2I_{xx}$, since $I_{xx} = I_{yy}$ by symmetry

$$\therefore \quad I_{xx} = \frac{1}{2}I_{zz} = \frac{\pi d^4}{64}.$$

(b) Hollow circular section: external diameter D, internal diameter d.

$$I = \frac{\pi}{64}D^4 - \frac{\pi}{64}d^4$$

$$= \frac{\pi}{64}(D^4 - d^4)$$

$$Z = \frac{I}{y_{max}} = \frac{\pi}{64}(D^4 - d^4) \div \frac{D}{2} = \frac{\pi\,(D^4 - d^4)}{32D}$$

$$I = A.k^2 \quad \therefore \quad \frac{\pi}{64}(D^4 - d^4) = \frac{\pi}{4}(D^2 - d^2)\,k^2$$

$$\therefore \quad k^2 = \frac{D^2 + d^2}{16}.$$

(iv) Triangular section: base b, height h.

We shall first calculate the moment of Inertia of the section about its base AB. Divide the section into a series of strips parallel to the base and consider an elementary strip of thickness δy at a distance y from AB. If b_x is the width of the strip,

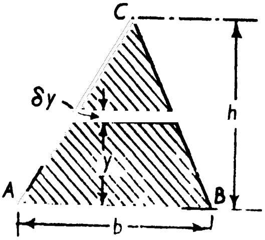


Fig. 80

$$\frac{b_x}{b} = \frac{h - y}{h}$$

$$\therefore \quad b_x = (h - y)\frac{b}{h}.$$

The area of the strip $\delta a = b_x.\delta y = \frac{b}{h}(h - y)\ \delta y.$

$$I_{AB} = \Sigma y^2.\delta a = \frac{b}{h}\int_0^h y^2(h - y)\ dy = \frac{b}{h}\left(h.\frac{y^3}{3} - \frac{y^4}{4}\right)_0^h$$

$$= \frac{b}{h}\left(\frac{h^4}{3} - \frac{h^4}{4}\right) = \frac{1}{12}\,bh^3.$$

The moment of Inertia I_{GG} about a parallel axis through the centre of gravity of the section can be obtained by the parallel-axes theorem, since,

$$I_{AB} = I_{GG} + \frac{1}{2}\,b.h.\left(\frac{h}{3}\right)^2$$

$$= I_{GG} + \frac{1}{18}\,bh^3$$

$$\therefore\quad I_{GG} = I_{AB} - \frac{bh^3}{18} = \frac{bh^3}{12} - \frac{bh^3}{18} = \frac{bh^3}{36}.$$

Example 6.

A floor has to carry a load of 160 lb. per sq. foot. If the span of each joist which is 3″ wide × 9″ deep is 15 feet, calculate the spacing centre to centre if the maximum permissible bending stress is 1200 lb. per sq. inch.

The section-modulus of each joist $= \frac{1}{6}bd^2 = \frac{1}{6} \times 3 \times 81$

$= 40{\cdot}5$ in^3.

The moment of resistance of each joist at a maximum stress of 1200 lb/in^2., is given by,

$M_r = f.Z = 1200 \times 40{\cdot}5 = 48600$ lb-inches.

If the joists are pitched x feet apart centre to centre, the total load per joist, $W = 160 \times x \times 15 = 2400x$ lb-wt. The maximum B.M. at the central section,

$$= \frac{Wl}{8} = \frac{2400 \times x \times 15}{8}$$

$= 4500x$ lb-feet.

$= 54000x$ lb-inches.

Equating,

$54000x = 48600$

$\therefore\quad x = {\cdot}9$ feet, say $10\frac{1}{2}''$ centre to centre.

Example 7.

Find the dimensions of a timber joist, span 20 feet, to carry a brickwall 9″ thick and 10 feet high, if the weight of brickwork is 120 lb. per cub. foot and the maximum permissible stress is limited to 1100 lb/in^2. The depth of the joist is to be twice its width.

The weight of the brickwall $W = 120 \times \frac{3}{4} \times 10 \times 20$

$= 18000$ lb-wt.

$$\text{Maximum B.M.} = \frac{Wl}{8} \times 12 \text{ lb-inches.}$$

$$= \frac{18000 \times 20 \times 12}{8} \text{ or } 540000 \text{ lb-inches.}$$

If b is the width and d the depth of the joist in inches,

$$Z = \tfrac{1}{6}\, bd^2.$$

The moment of resistance at 1100 lb/in². is,

$$M_r = 1100 \times \tfrac{1}{6}\, bd^2 \text{ lb-inches.}$$

Equating the moment of resistance to the B.M.,

$$\frac{1100}{6} bd^2 = 540000$$

$$\therefore \quad bd^2 = \frac{32400}{11}$$

Since, $d = 2b$, we get,

$$4b^3 = \frac{32400}{11}$$

$$\therefore \quad b^3 = 736{\cdot}36$$

or $\quad b = 9{\cdot}03''$

and $\quad d = 18{\cdot}06''$.

A section 9″ × 18″ will therefore do.

Example 8.

A 12″ × 5″ rolled steel joist of I-section has flanges ·55″ thick and web ·35″ thick. If the permissible skin stress is limited to 7½ tons/in²., find the safe uniformly distributed load that this section will carry over a span of 16 feet.

I_{xx} for the section $= \frac{1}{12}(BD^3 - bd^3)$, where $B = 5''$,

$D = 12''$, $b = (5 - {\cdot}35)$ or $4{\cdot}65''$ and

$d = (12 - 2 \times {\cdot}55) = 10{\cdot}9''$.

$$I_{xx} = \tfrac{1}{12}(5 \times 12^3 - 4{\cdot}65 \times 10{\cdot}9^3) = \tfrac{1}{12}(8640 - 6022)$$

$$= 218{\cdot}2 \text{ in}^4.$$

$$Z = \frac{218{\cdot}2}{6} = 36{\cdot}37 \text{ in}^3.$$

The moment of resistance at 7½ tons/in². is,

$$M_r = f.Z = 7{\cdot}5 \times \frac{218{\cdot}2}{6} = 272{\cdot}75 \text{ ton-inches.}$$

If W is the total uniformly distributed load in tons,

$$M_{\max} = \frac{Wl}{8} \times 12 \text{ ton-inches}$$

$$= \frac{W \times 16 \times 12}{8} \text{ or } 24W \text{ ton-inches.}$$

Equating, $24W = 272{\cdot}75$

$\therefore \quad W = 11{\cdot}37$ tons

or $\quad w = {\cdot}71$ ton per foot run.

Example 9.

A cast iron water pipe 18″ bore, $\frac{3}{4}$″ thick, is supported at two points 30 feet apart. Find the maximum stress in the metal when it is running full. Density of cast iron 450 lb. per cub. foot and of water 62·5 lb. per cub. foot.

The section of the pipe is a hollow circular section,

$D = 19{\cdot}5''$; $d = 18''$.

$$\text{The area of section } A = \frac{\pi}{4}(19{\cdot}5^2 - 18^2) = \frac{\pi}{4} \times 37{\cdot}5 \times 1{\cdot}5$$

$$= 44{\cdot}18 \text{ sq. in} = {\cdot}3068 \text{ sq. ft.}$$

$$\text{Dead weight of pipe} = 450 \times {\cdot}3068 \times 30 = 4142 \text{ lb-wt.}$$

$$\text{Weight of water} = 62{\cdot}5 \times \frac{\pi}{4} \times 1{\cdot}5^2 \times 30 = 3314 \text{ lb-wt.}$$

$$\therefore \quad \text{Total weight } W = 7456 \text{ lb-wt.}$$

$$= 3{\cdot}33 \text{ ton-wt.}$$

$$\text{Maximum B.M.} = \frac{Wl}{8} \times 12 \text{ ton-inches.}$$

$$= \frac{3{\cdot}33 \times 30 \times 12}{8} \text{ or } 150 \text{ ton-inches.}$$

$$I_{xx} \text{ for the section} = \frac{\pi}{64}(D^4 - d^4) = \frac{\pi}{64}(19{\cdot}5^4 - 18^4).$$

$$= \frac{\pi}{64}(380{\cdot}2 + 324)(380{\cdot}2 - 324)$$

$$= \frac{\pi}{64} \times 704{\cdot}2 \times 56{\cdot}2 = 1942 \text{ in}^4.$$

$$\therefore \qquad Z_{xx} = \frac{1942}{9\cdot75} = 199\cdot2 \text{ in}^3.$$

The maximum stress $f = \frac{M_r}{Z} = \frac{150}{199\cdot2} = \cdot753 \text{ ton/in}^2.$

8. Strength of sections: The strength of a beam section does not depend upon the sectional area provided but upon the disposition of that area in relation to its neutral axis, in other words, on the section-modulus Z. A beam of rectangular section b wide and d deep has a modulus of section of $\frac{1}{6}\, bd^2$. The same section, used as a beam with d as width and b as depth has a modulus of section of only $\frac{1}{6}\, db^2$. Used in this manner, the strength of the beam is reduced in the proportion $\frac{1}{6}db^2 \div \frac{1}{6}bd^2$ or $\frac{b}{d}$. If d is equal to $2b$, the reduction in strength is 50%.

Example 10.

For a given stress, compare the moments of resistance of a beam of suqare section placed (i) with two sides horizontal and (ii) with a diagonal horizontal.

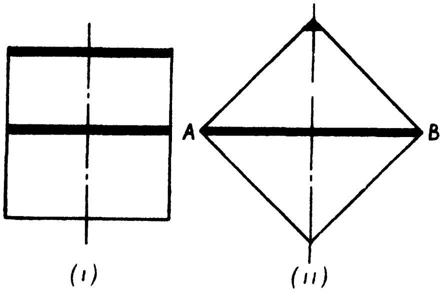


FIG. 81

Let b be the side of the square section. When placed with two sides horizontal, the modulus of section $Z_1 = \frac{1}{6}\, b.b^2 = \frac{1}{6}\, b^3$.

In the second case with a horizontal diagonal, the moment of Inertia of section is,

$$I_{AB} = 2 \times \tfrac{1}{12} \times b\sqrt{2} \times (\tfrac{1}{2}b\sqrt{2})^3$$
$$= \frac{b^4}{12}.$$

$$y_{max} = \frac{b}{\sqrt{2}}.$$

$\therefore$ The modulus of section $Z_2 = \frac{I_{AB}}{y_{max}} = \frac{b^4}{12} \div \frac{b}{\sqrt{2}} = \frac{b^3}{6\sqrt{2}}$

The ratio of strengths $= \frac{Z_1}{Z_2} = \sqrt{2} = 1{\cdot}414.$

The square section is thus 41·4% stronger if placed with two sides horizontal than with a diagonal horizontal.

This is evident from a glance at the section. If it is divided into a series of thin strips parallel to the N.A., it is seen in the second case that maximum area is provided at the N.A. where the stress is zero and minimum area, furthest away from the N.A. where it should be most effective. The disposition of the sectional area is thus wasteful in case (ii) as compared with case (i), where the same elementary area of strip is available from the N.A. upto the extremities of the section.

Example 11.

A beam of I-section, 12″ × 6″ has flanges $\frac{7}{8}$″ thick and web $\frac{1}{2}$″ thick. Compare its flexural strength with that of a beam of rectangular section of the same weight, the depth being twice the width.

For the *I*-section,

$$I_{xx} = \frac{1}{12}[6 \times 12^3 - 5{\cdot}5 \times (10{\cdot}25)^3]$$
$$= 864 - 493{\cdot}5 = 370{\cdot}5 \text{ in}^4.$$

The modulus of section $Z_1 = \frac{370{\cdot}5}{6} = 61{\cdot}75 \text{ in}^3.$

For a rectangular beam of the same weight, the area of section must be the same.

$$\text{Area} = 2 \times 6 \times \tfrac{7}{8} + \tfrac{1}{2} \times 10{\cdot}25$$
$$= 10{\cdot}5 + 5{\cdot}125 = 15{\cdot}625 \text{ in}^2.$$

Since the depth is to be twice the width b, we get,

$$2b^2 = 15{\cdot}625$$
$$\therefore \quad b^2 = 7{\cdot}8125$$

$$\text{or} \quad b = 2{\cdot}8''$$

$$\therefore \quad d = 5{\cdot}6''.$$

The modulus of this section $Z_2 = \frac{1}{6} \times 2{\cdot}8 \times 5{\cdot}6^2$

$$= 14{\cdot}63 \text{ in}^3.$$

$$\text{The ratio of strengths} = \frac{Z_1}{Z_2} = \frac{61{\cdot}75}{14{\cdot}63} = 4{\cdot}22.$$

The beam of *I*-section is thus more than four times as strong as a beam of rectangular section of the same area.

From this example, it is seen that even a rectangular section is wasteful as compared to the *I*-section, in which the disposition of area is maximum in the flanges where it is most effective, the minimum being arranged in the web to connect the flanges. For timber beams, the usual section is a rectangle but for mild steel, the maximum possible use is made of the sectional area by adopting the *I*-section.

Example 12.

In the preceding example, compare the flexural strength of the two sections with that of a solid circular section of the same area.

If a circular section of the same area is provided, its diameter d will be given by,

$$\frac{\pi}{4} d^2 = 15{\cdot}625$$

$$\therefore \quad d^2 = 19{\cdot}89$$

$$\text{or} \quad d = 4{\cdot}46''.$$

The modulus of section Z_3 in this case is only $\frac{\pi}{32} d^3$

i.e. $8{\cdot}71$ in^3.

If the strength of this section is taken as unity, that of the rectangular section is $\frac{14{\cdot}63}{8{\cdot}71}$ or $1{\cdot}68$ and of the *I*-section it is $\frac{61{\cdot}75}{8{\cdot}71}$ or $7{\cdot}09$.

The disposition of the sectional area in the three cases is shown in fig. 82 from which it is clear that a circular section is very uneconomical for a beam section.

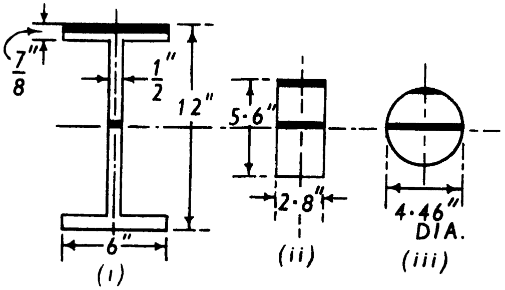


FIG. 82

9. British Standard Sections: Mild steel is rolled into standard sections such as the *I*-section, channel, Tee, angle with equal and unequal legs and *Z*-sections and a wide variety of sizes is available for use as beams and struts. The most important of these is the *I*-section which is available in thirty standard sizes ranging from 24" × 7½" × 100 lb. per foot run to 3" × 1½" × 4 lb. per foot run. Tables giving the dimensions and properties of standard sections are published by manufacturers and reference should be made to their pocket books or any standard work on Theory of Structures.

10. The I-Section: The I_{xx} of this section has been shown to be $\frac{1}{12}(BD^3 - bd^3)$ and the modulus of section Z is therefore $\dfrac{BD^3 - bd^3}{6D}$. A simpler expression for the modulus of section can be derived on the assumption that the entire bending moment is resisted by the flanges i.e. the resistance of the web is neglected. If f is the intensity of maximum stress and f_1 that at the junction with the web, the average intensity of stress across the flange-area is $\dfrac{f+f_1}{2}$. The total push C of the top flange = average stress × area of flange $= (\dfrac{f+f_1}{2}).A$; f_1 is a little less than f and if we assume that the average stress is for all practical purposes equal to f, the total thrust $C = f.A$. Similarly the total pull T of the bottom flange $= f.A$. These two forces form the resisting couple or moment of resistance, the arm of the couple being the distance between the centres of gravity of the trapezium-shaped stress intensity diagrams for flanges. If we take the arm of the couple as the distance between the centres of flanges, the assumption will be on the safe side.

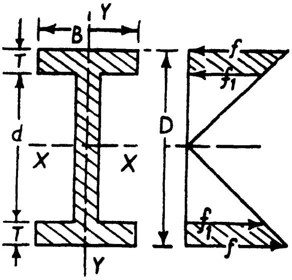


FIG. 83

The moment of resistance M_r is therefore $= C$ or $T \times a$ $= fA \times$ distance between centres of flanges. For purposes of design we may even take the arm a equal to the depth d of the web.

The formula then becomes,

$$M_r = f.A.d$$

$= f.B.T.d$ where B and T are the width and thickness of the flange and d is the depth of the web. The modulus of section of the I-section is then simply $A \times d$ i.e. the product of the area of one flange and the depth of the web. This approximation becomes particularly useful when the I-section has to be built up as in the case of plate-girders.

11. Moments of Inertia of unsymmetrical sections:

Example 13.

The thickness of flanges and web of an 8″ × 3″ British Standard Channel-section are ½″ and ⅜″ respectively. Find the position of the c.g. of the section and its I_{xx} *and* I_{yy}.

The section is symmetrical about the axis XX. Let P be the distance of its c.g. from the base AB. Split up the section into three areas:

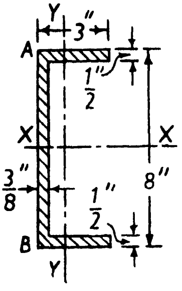


FIG. 84

(i) Top-flange: area $a_1 = 3 \times \frac{1}{2} = 1{\cdot}5$ in^2.; x_1 the distance of the c.g. from $AB = \frac{1}{2} \times 3 = \frac{3}{2}''$.

(ii) Web: area $a_2 = (8 - 2 \times \frac{1}{2}) \times \frac{3}{8} = \frac{21}{8}$ in^2.;
$x_2 = \frac{1}{2} \times \frac{3}{8} = \frac{3}{16}''$.

(iii) Bottom flange: $a_3 = 3 \times \frac{1}{2} = \frac{3}{2}$ in^2.; $x_3 = \frac{3}{2}''$.

Total area of section $a = a_1 + a_2 + a_3 = \dfrac{45}{8} = 5{\cdot}625$ in^2.

Taking moments about AB,

$$P = \frac{a_1x_1 + a_2x_2 + a_3x_3}{a_1 + a_2 + a_3} = \frac{\frac{3}{2} \times \frac{3}{2} + \frac{21}{8} \times \frac{3}{16} + \frac{3}{2} \times \frac{3}{2}}{\frac{45}{8}}$$
$$= \frac{71}{80} = {\cdot}888''.$$

$$I_{xx} \text{ for the section } = \tfrac{1}{12}\,(BD^3 - bd^3)$$
$$= \tfrac{1}{12}\,(3 \times 8^3 - 2\tfrac{5}{8} \times 7^3) = 128 - 75{\cdot}03$$
$$= 52{\cdot}97 \text{ in}^4.$$

To get the I_{yy}, we shall first obtain I_{AB} which is the sum of the moments of Inertia of three rectangles comprising the flanges and the web about a side.

$$I_{AB} = \tfrac{1}{3} \times \tfrac{1}{2} \times 3^3 + \tfrac{1}{3} \times 7 \times (\tfrac{3}{8})^3 + \tfrac{1}{3} \times \tfrac{1}{2} \times 3^3$$
$$= 9{\cdot}123 \text{ in}^4.$$

([illegible])

Using the parallel-axes theorem

$$I_{AB} = I_{yy} + a.P^2$$
$$\therefore \quad I_{yy} = I_{AB} - aP^2$$
$$= 9{\cdot}123 - 5{\cdot}625 \times ({\cdot}888)^2$$
$$= 9{\cdot}123 - 4{\cdot}435 = 4{\cdot}688 \text{ in}^4.$$

Example 14.

Find the position of the c.g. and calculate I_{xx} and I_{yy} for a 6" × 4" × $\frac{5}{8}$" British Standard Tee-section.

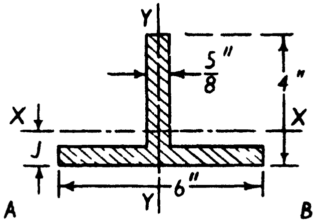


FIG. 85

The section is symmetrical about the axis *YY*. Let J be the distance of its c.g. from the base *AB*. Divide the section into two rectangles:

(i) The web portion upto *AB*:
Area $a_1 = 4 \times \frac{5}{8} = \frac{5}{2}$ in^2.;
the distance of its c.g. from *AB* is $y_1 = 2''$.

(ii) The remaining flange portion:

$$\text{Area } a_2 = (6 - \frac{5}{8})\,\frac{5}{8} = \frac{43}{8} \times \frac{5}{8} = \frac{215}{64} \text{ in}^2.$$

$$y_2 = \frac{5''}{16}.$$

Taking moments about *AB*,

$$J = \frac{a_1 y_1 + a_2 y_2}{a_1 + a_2} = \frac{\frac{5}{2} \times 2 + \frac{215}{64} \times \frac{5}{16}}{\frac{5}{2} + \frac{215}{64}} = 1{\cdot}03''$$

$I_{AB} = \frac{1}{3} \times \frac{5}{8} \times 4^3 + \frac{1}{3}(6 - \frac{5}{8})\,(\frac{5}{8})^3 = 13{\cdot}33 + {\cdot}44 = 13{\cdot}77$ in^4.

$I_{AB} = I_{xx} + a.J^2$, by the parallel-axes theorem.

$\therefore \quad I_{xx} = I_{AB} - a.J^2 = 13{\cdot}77 - 5{\cdot}9 \times (1{\cdot}03)^2$

$= 13{\cdot}77 - 6{\cdot}25 = 7{\cdot}52$ in^4.

To obtain I_{yy}, take the flange and the web separately;

$I_{yy} = \frac{1}{12} \times \frac{5}{8} \times 6^3 + \frac{1}{12}(4 - \frac{5}{8})\,(\frac{5}{8})^3 = 11{\cdot}25 + {\cdot}07 = 11{\cdot}32$ in^4.

Example 14a.

$6'' \times 4'' \times \frac{5}{8}''$ Tee-joists are pitched 18″ centre to centre to span a room 10 feet wide. If the floor load is $1\frac{1}{2}$ cwt. per square foot, calculate the maximum stresses induced in the section. Neglect the weight of the joist.

The load taken by each joist is,

$$W = \frac{3}{40} \times \frac{3}{2} \times 10 = \frac{9}{8} \text{ tons.}$$

The maximum B.M. at the central section

$$= \frac{Wl}{8} \times 12, \text{ ton-in.} = \frac{9}{8} \times \frac{10 \times 12}{8} = \frac{135}{8}$$

$$= 16{\cdot}875 \text{ ton-inches.}$$

Referring to figure 85, $y_c = 2{\cdot}97''$; $y_t = 1{\cdot}03''$; $I_{xx} = 7{\cdot}52 \text{ in}^4$.

$$M_r = \frac{f_c}{y_c} \times I = \frac{f_t}{y_t} \times I = 16{\cdot}875$$

$$\therefore \quad f_c = \frac{16{\cdot}875 \times 2{\cdot}97}{7{\cdot}52} = 6{\cdot}65 \text{ tons/in}^2.$$

$$f_t = \frac{16{\cdot}875 \times 1{\cdot}03}{7{\cdot}52} = 2{\cdot}31 \text{ tons/in}^2.$$

Example 15.

Find the position of the c.g. and I_{xx} and I_{yy} for an unequal angle section $5'' \times 4'' \times \frac{1}{2}''$.

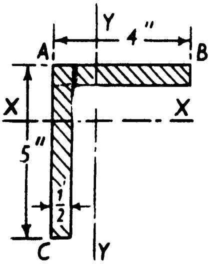


FIG. 86

Split up the section into two rectangles:

(i) $a_1 = 5 \times \frac{1}{2} = 2{\cdot}5 \text{ in}^2.$; $x_1 = \frac{1}{4}''$; $y_1 = \frac{5}{2}''$, using AB and AC as axes of reference.

(ii) $a_2 = (4 - \frac{1}{2}) \times \frac{1}{2} = 1{\cdot}75 \text{ in}^2$; $x_2 = \frac{1}{2} + \frac{1}{2} \times 3\frac{1}{2} = \frac{9}{4}''$
$y_2 = \frac{1}{4}''$

$$\bar{x} = \frac{a_1x_1 + a_2x_2}{a_1 + a_2} = \frac{\frac{5}{2} \times \frac{1}{4} + \frac{7}{4} \times \frac{9}{4}}{\frac{17}{4}} = \frac{73}{68} = 1{\cdot}073''$$

$$\bar{y} = \frac{a_1y_1 + a_2y_2}{a_1 + a_2} = \frac{\frac{5}{2} \times \frac{5}{2} + \frac{7}{4} \times \frac{1}{4}}{\frac{17}{4}} = \frac{107}{68} = 1{\cdot}57''$$

$$I_{AB} = \tfrac{1}{3} \times \tfrac{1}{2} \times 5^3 + \tfrac{1}{3} \times 3\tfrac{1}{2} \times (\tfrac{1}{2})^3 = 20{\cdot}83 + {\cdot}146 = 20{\cdot}976 \text{ in}^4.$$

$$I_{AB} = I_{xx} + a.\bar{y}^2$$

$$\therefore \quad I_{xx} = I_{AB} - a.\bar{y}^2 = 20{\cdot}976 - \tfrac{17}{4}\ (1{\cdot}57)^2.$$

$$= 20{\cdot}976 - 10{\cdot}5 = 10{\cdot}476 \text{ in}^4.$$

For I_{yy}, divide the section into two rectangles $4'' \times \frac{1}{2}''$ and $4\frac{1}{2}'' \times \frac{1}{2}''$. Then,

$$I_{AC} = \tfrac{1}{3} \times \tfrac{1}{2} \times 4^3 + \tfrac{1}{3} \times 4\tfrac{1}{2}\ (\tfrac{1}{2})^3 = 10{\cdot}67 + {\cdot}19 = 10{\cdot}86 \text{ in}^4.$$

$$I_{AC} = I_{yy} + a.\bar{x}^2$$

$$\therefore \quad I_{yy} = I_{AC} - a.\bar{x}^2 = 10{\cdot}86 - \tfrac{17}{4}\ (1{\cdot}073)^2$$

$$= 10{\cdot}86 - 4{\cdot}89 = 5{\cdot}97 \text{ in}^4.$$

Example 15a.

A 5″ × 4″ × ½″ unequal angle bar is placed with the longer leg vertical and used as a joist freely supported at the ends. Find what uniformly distributed load will be carried by the joist over a span of 12 feet if the maximum bending stress is not to exceed 7 tons/in².

$I_{xx} = 10{\cdot}476$ in⁴.; $y_c = 1{\cdot}57''$; $y_t = 3{\cdot}43''$.

Since $\dfrac{f_c}{y_c} = \dfrac{f_t}{y_t}$ or $f_c = \dfrac{y_c}{y_t} \times f_t$, the maximum permissible stress of 7 tons/in². will be reached on the tension side. If f_c is allowed to go up to 7 tons/in²., f_t will be just over 15 tons/in²., which is not permissible; whereas if $f_t = 7$ tons/in²., the corresponding $f_c = \dfrac{1{\cdot}57}{3{\cdot}43} \times 7 = 3{\cdot}2$ tons/in²., which is allright.

The moment of resistance $M_r = f_t Z_t = 7 \times \dfrac{10{\cdot}476}{3{\cdot}43}$

$$= 21{\cdot}38 \text{ ton-inches.}$$

If W is the total uniformly distributed load over the span,

$$M_{\max} = \frac{Wl}{8} \times 12 \text{ ton-inches} = \frac{W \times 12 \times 12}{8} \quad \text{or} \quad 18W$$
ton-inches.

Equating the B.M. to the moment of resistance,

$$18W = 21{\cdot}38$$

$$\therefore \quad W = 1{\cdot}19 \text{ tons.}$$

Note: The result is approximate; actually, it is a case of unsymmetrical bending, beyond the scope of this book.

Example 16.

A cast iron beam of I-section has the top flange 6" × 1" thick, web 9" × 1" and the bottom flange 10" × 2" thick. Find the c.g. of the section and calculate the I_{xx} and I_{yy} of the section.

Take *AB* the bottom side of the section as axis of reference. The section can be split up into three rectangles.

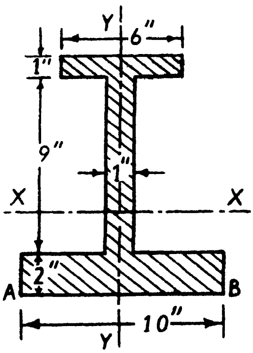


FIG. 87

(i) Bottom flange:
Area $a_1 = 10 \times 2 = 20$ in^2; $y_1 = \frac{1}{2} \times 2 = 1''$.

(ii) The web:
$a_2 = 9 \times 1 = 9$ in^2.; $y_2 = 2 + \frac{1}{2} \times 9 = \frac{13}{2}''$.

(iii) Top flange:
$a_3 = 6 \times 1 = 6$ in^2.; $y_3 = 2 + 9 + \frac{1}{2} \times 1 = \frac{23}{2}''$.

$$\bar{y} = \frac{a_1y_1 + a_2y_2 + a_3y_3}{a_1 + a_2 + a_3} = \frac{20 \times 1 + 9 \times \frac{13}{2} + 6 \times \frac{23}{2}}{35}$$

$$= \frac{147{\cdot}5}{35} = 4{\cdot}214''.$$

$$I_{AB} = [\tfrac{1}{3} \times 10 \times 2^3] + [\tfrac{1}{12} \times 1 \times 9^3 + 9 \times (\tfrac{13}{2})^2] + [\tfrac{1}{12} \times 6 \times 1^3 + 6 \times (\tfrac{23}{2})^2]$$

$$= 26{\cdot}67 + 60{\cdot}75 + 380{\cdot}25 + {\cdot}50 + 793{\cdot}50$$
$$= 1261{\cdot}67 \text{ in}^4.$$
$$\bar{I}_{xx} = I_{xx} + a.\bar{y}^2 = I_{xx} + 35 \times 4{\cdot}214^2.$$
$$I_{xx} = 1261{\cdot}67 - 621{\cdot}6 = 640{\cdot}07 \text{ in}^4.$$
$$I_{yy} = \tfrac{1}{12}(2 \times 10^3 + 9 \times 1^3 + 1 \times 6^3) = \tfrac{2225}{12} = 185{\cdot}42 \text{ in}^4.$$

Example 16a.

A C.I. beam of the section given above is supported over a span of 20 feet. If the permissible stresses are 6 tons/in². in compression and 1½ tons/in². in tension, what uniformly distributed load will the beam safely carry?

I_{xx} of the beam = 640 in⁴.

$$y_c = 7{\cdot}786'';$$
$$y_t = 4{\cdot}214''$$
$$\frac{f_c}{y_c} = \frac{f_t}{y_t}$$
$$\therefore \qquad f_c = \frac{y_c}{y_t} \times f_t = \frac{7{\cdot}786}{4{\cdot}214} \times f_t.$$

When f_t reaches 1·5 tons/in²., the corresponding stress f_c is $\frac{7{\cdot}786}{4{\cdot}214} \times 1{\cdot}5$ or 2·77 tons/in²., which is within the permissible limit of 6 tons/in². Should f_c be allowed to go to 6 tons/in²., the corresponding value of f_t will evidently exceed the permissible limit of 1½ tons/in². The permissible stresses are therefore $f_c = 2{\cdot}77$ tons/in². and $f_t = 1{\cdot}5$ tons/in².

$$M_r = f_c Z_c = f_t Z_t = \frac{2{\cdot}77 \times 640}{7{\cdot}786} \text{ or } \frac{1{\cdot}5 \times 640}{4{\cdot}214}$$
$$= 227{\cdot}7 \text{ tons-inches.}$$

If W is the total uniformly distributed load in tons,

$$M_{\max} = \frac{Wl}{8} \times 12 \text{ ton-inches}$$
$$= \frac{W \times 20 \times 12}{8} \text{ or } 30W \text{ ton-inches.}$$

Equating, $30W = 227{\cdot}7$

$\therefore \qquad W = 7{\cdot}59$ tons

and $\qquad w = \frac{7{\cdot}59}{20} = {\cdot}38$ ton per foot run.

12. Moments of Inertia of built-up sections: A variety of compound sections is built up of standard channel, *I*, or angle sections and flat plates to make beam and column sections of suitable strength and stiffness. The dimensions and properties of standard sections are available from tables. The moments of Inertia of built-up sections are then easily obtained by the use of these tables and the parallel-axes theorem. A few of these built-up sections are shown in fig. 88.

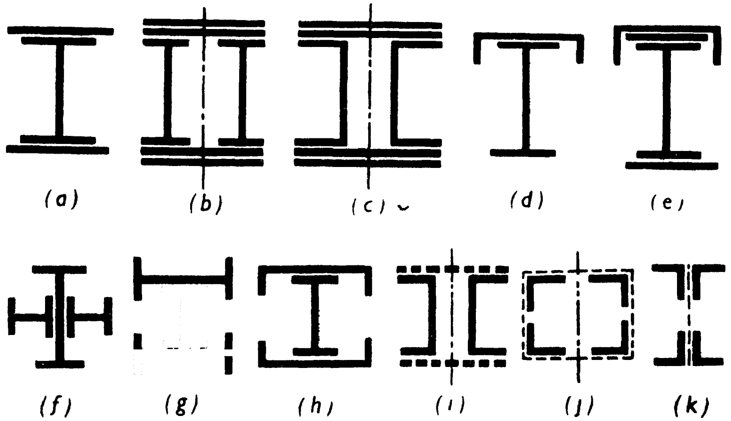


FIG. 88

Example 17.

A compound section is built up of two 12″ × 6″ × 44 lb. R.S. beams placed side by side with two $\frac{1}{2}''$ *plates 14″ wide riveted to each top and bottom flange. Calculate the* I_{xx} *and* I_{yy} *of the section. For each 12″ × 6″ × 44 lb. joist, area of section* $A = 12{\cdot}94$ *in*^{2}.; $I_{xx} = 315{\cdot}3$ *in*^{4}; $I_{yy} = 22{\cdot}27$ *in*^{4}.

The section is symmetrical. Referring to fig. 88(*b*), the I_{xx} of the built-up section will be that for the two joists plus the moments of inertia of two rectangular areas 14″ × 1″ about the axis *XX* of symmetry.

Using the parallel-axes theorem, the moment of Inertia of each 14″ × 1″ rectangular area about *XX*

$$= \tfrac{1}{12} \times 14 \times 1^3 + 14 \times 1 \times (6 + \tfrac{1}{2})^2$$

$$= 1{\cdot}17 + 591{\cdot}5 = 592{\cdot}67 \text{ in}^4.$$

The moment of Inertia of the built-up section is therefore,

$$I_{xx} = 2 \times 315{\cdot}3 + 2 \times 592{\cdot}67$$
$$= 630{\cdot}6 + 1185{\cdot}33$$
$$= 1815{\cdot}93 \text{ in}^4.$$

Similarly,

$$I_{yy} = 2\ [22{\cdot}27 + 12{\cdot}94 \times 3^2] + 2\ [\tfrac{1}{12} \times 1 \times 14^3]$$
$$= 44{\cdot}54 + 232{\cdot}92 + 457{\cdot}33$$
$$= 734{\cdot}79 \text{ in}^4.$$

Example 18.

A built-up section is made up of two standard 15" × 4" R.S. Channels spaced $7\frac{1}{2}$" apart back to back, with four 16" × $\frac{1}{2}$" plates riveted, two at the top and two at the bottom flange. Find the I_{xx} and I_{yy} of the built-up section. For each 15" × 4" Channel, $A = 12{\cdot}33$ in^2.; $P = {\cdot}935$"; $I_{xx} = 377$ in^4.; $I_{yy} = 14{\cdot}55$ in^4.

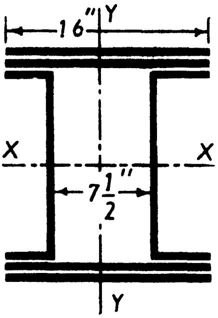


FIG. 89

$$I_{xx} = 2 \times 377 + 2\ [\tfrac{1}{12} \times 16 \times 1^3 + 16 \times 1 \times (7{\cdot}5 + {\cdot}5)^2]$$
$$= 754 + 2{\cdot}67 + 2048 = 2804{\cdot}67 \text{ in}^4.$$

$$I_{yy} = 2\ [14{\cdot}55 + 12{\cdot}33\ (\frac{7{\cdot}5}{2} + {\cdot}935)^2] + 2 \times \tfrac{1}{12} \times 1 \times 16^3$$
$$= 29{\cdot}10 + 541{\cdot}3 + 682{\cdot}67 = 1253{\cdot}07 \text{ in}^4.$$

Example 19.

A column is built-up of three rolled steel joists in the form of a cross as shown in fig. 88(f). The bigger joist is 16"×6"×50 lb. The smaller joists attached to its web on either side are 9"×7"×50 lb. The properties of the sections are:

For the 16″×6″ joist, $A = 14{\cdot}71$ *in*^{2}.; $I_{xx} = 618{\cdot}02$ *in*^{4}.; $I_{yy} = 22{\cdot}47$ *in*^{4}.; *thickness of web* = ·4″.
For the 9″×7″ joist, $A = 14{\cdot}71$ *in*^{2}.; $I_{xx} = 208{\cdot}13$ *in*^{4}.; $I_{yy} = 40{\cdot}17$ *in*^{4}.

Calculate the I_{xx} *and* I_{yy} *of the compound section and the respective radii of gyration.*

$$I_{xx} = 618{\cdot}09 + 2 \times 40{\cdot}17 = 698{\cdot}43 \text{ in}^4.$$
$$I_{yy} = 22{\cdot}47 + 2\,[208{\cdot}13 + 14{\cdot}71 \times (4{\cdot}5 + \tfrac{1}{2} \times {\cdot}4)^2]$$
$$= 22{\cdot}47 + 416{\cdot}26 + 650 = 1088{\cdot}73 \text{ in}^4.$$

Total area of section $= 14{\cdot}71 \times 3 = 44{\cdot}13$ in^2.

$$I_{xx} = 698{\cdot}43 = 44{\cdot}13\,k_{xx}^2 \quad \therefore \quad k_{xx}^2 = 15{\cdot}8 \text{ or } k_{xx} = 3{\cdot}98''$$
$$I_{yy} = 1088{\cdot}73 = 44{\cdot}13\,k_{yy}^2 \quad \therefore \quad k_{yy}^2 = 24{\cdot}7 \text{ or } k_{yy} = 4{\cdot}97''.$$

Example 20.

A braced crane-jib is built-up of four $3\tfrac{1}{2}'' \times 3\tfrac{1}{2}'' \times \tfrac{3}{8}''$ *angles forming a square section 18″ overall as in fig. 88(j). Calculate the moment of Inertia and the radius of gyration of the section about either axis of symmetry. For the* $3\tfrac{1}{2}'' \times 3\tfrac{1}{2}'' \times \tfrac{3}{8}''$ *angle,* $A = 2{\cdot}49$ *in*^{2}.; $J = 1''$; $I_{xx} = I_{yy} = 2{\cdot}80$ *in*^{4}.

$$I_{xx} = 4\;[2{\cdot}80 + 2{\cdot}49\;(9-1)^2]$$
$$= 4\;[2{\cdot}80 + 159{\cdot}36] = 4 \times 162{\cdot}16 = 648{\cdot}64 \text{ in}^4.$$
$$\therefore \quad k_{xx}^2 = \frac{648{\cdot}64}{4 \times 2{\cdot}49} = 65{\cdot}2$$
$$\therefore \quad k_{xx} = 8{\cdot}1''.$$

Example 21.

A latticed stanchion is made up of two 12″×4″ standard channels placed back to back at a distance x inches apart as shown in fig. 88(i). What should be the value of x so that the stanchion is equally strong in all directions? For the 12″×4″ channel, $A = 9{\cdot}21$ *in*^{2}.; $P = 1{\cdot}055''$; $I_{xx} = 200{\cdot}1$ *in*^{4}.; $I_{yy} = 12{\cdot}12$ *in*^{4}.

For equal strength in all directions, $I_{xx} = I_{yy}$.

$$I_{xx} = 2 \times 200{\cdot}1 \text{ and } I_{yy} = 2\,[12{\cdot}12 + 9{\cdot}21 \times (\frac{x}{2} + 1{\cdot}055)^2].$$

Equating,

$$12{\cdot}12 + 9{\cdot}21\;(\frac{x}{2} + 1{\cdot}055)^2 = 200{\cdot}1$$

$$\therefore \quad (\frac{x}{2} + 1{\cdot}055)^2 = \frac{187{\cdot}98}{9{\cdot}21} = 20{\cdot}42$$

$$\therefore \quad \frac{x}{2} + 1{\cdot}055 = 4{\cdot}52$$

$$\therefore \quad \frac{x}{2} = 3{\cdot}465'' \text{ or } x = 6{\cdot}93'', \text{ say } 7''.$$

13. Graphical method for moment and moment of Inertia of a plane section about an axis in its plane: Let a be the area whose moment and moment of Inertia about the axis AB in its plane are required. Draw A_1B_1 parallel to AB at a distance h from it.

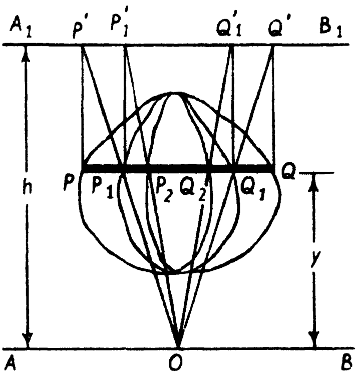


FIG. 90

Divide the area a into a series of thin strips parallel to AB. Consider a strip PQ of thickness δy at a distance y from AB. Let $P'Q'$ be the projection of PQ on A_1B_1. Select a pole O in AB and join OP' and OQ' cutting PQ at P_1Q_1. For every strip PQ, there will be a corresponding pair of points P_1, Q_1. As the area a is bounded by the points P and Q, the points P_1 and Q_1 will enclose an area which we shall call a_1. Then, the moment of the area a about $AB = a_1 \times h$.

Repeat the same operation on the area a_1 i.e. project P_1Q_1 to $P'_1Q'_1$ on A_1B_1. Join OP'_1 and OQ'_1, cutting P_1Q_1 at P_2Q_2. The points P_2 and Q_2 will enclose another area which may be called a_2.

The moment of Inertia of the area a about AB

$= a_2 \times h^2$.

Proof:

Referring to fig. 90,

$\frac{P_1Q_1}{PQ} = \frac{P_1Q_1}{P'Q'} = \frac{y}{h}$ from similar triangles OP_1Q_1 and $OP'Q'$.

$\therefore \quad y.PQ = h.P_1Q_1$.

Multiplying both sides by δy, we get,

$y.PQ.\ \delta y = h.P_1Q_1.\delta y$

or $\quad y.\delta a = h.\delta a_1$, δa and δa_1, being the areas of the elementary strips PQ and P_1Q_1 of thickness δy.

The moment of the area a about AB is the algebraic sum of the moments of elementary areas δa about AB.

$\therefore \quad a\bar{y} = \Sigma y.\delta a = \Sigma h.\delta a_1 = h\Sigma\,\delta a_1 = a_1 \times h$, which proves the proposition, $\bar{y}$ being the distance of the c.g. of the area a from AB.

Similarly,

$\frac{P_2Q_2}{P_1Q_1} = \frac{P_2Q_2}{P'_1Q'_1} = \frac{y}{h}$ from similar triangles OP_2Q_2 and $OP'_1Q'_1$.

$\therefore \quad P_2Q_2 = \frac{y}{h}.P_1Q_1 = \frac{y}{h}.\frac{y}{h}\,PQ = \frac{y^2}{h^2}.\,PQ$

$\therefore \quad y^2.PQ = h^2.P_2Q_2$.

Multiplying both sides by δy,

$y^2.\delta a = h^2.\delta a_2$, δa_2 being the area of elementary strip P_2Q_2 of thickness δy.

Now, $\quad I_{AB} = \Sigma y^2.\,\delta a = \Sigma h^2.\,\delta a_2 = h^2\Sigma\,\delta a_2 = a_2h^2$, which proves the proposition.

It will be noted that the purpose of the geometrical construction is to reduce the width of the strip in the ratio of $\frac{y}{h}$. By selecting a suitable pole O for each strip, it is possible to have the points P_1 and P_2 in common with P so that the areas a_1

and a_2 will have a portion of their boundaries in common with the perimeter of the area a as in fig. 91.

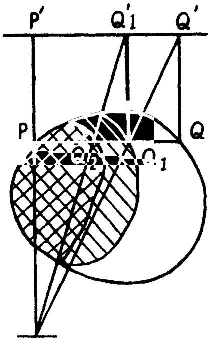


FIG. 91

The inner area a_1 is called the "Modulus figure" of the area a. It is extremely rare that one is required to obtain the moment of Inertia of any irregular figure by the graphical method. The "Modulus figure" however, has a significance which will now be noted.

14. The Modulus Figure: In a beam section we have seen that a part of the resisting area is under compression and a part under tension, the intensity of stress varying from zero at the neutral axis to a maximum at the extremities. It is possible to look upon this case of an area under a *varying* stress as equivalent to that of *uniform* stress on a modified area which is necessarily smaller than the original area. Consider a beam

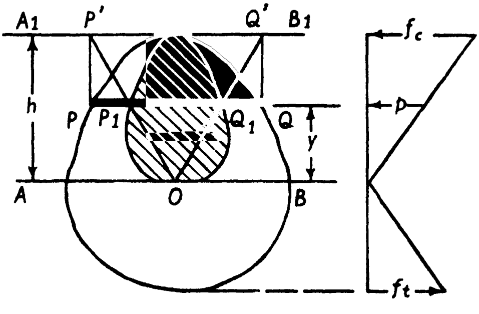


FIG. 92

section whose neutral axis is AB as shown in fig. 92. Let h be the distance of the compression edge from AB. Draw A_1B_1 parallel to AB through the compression edge. With any pole O on AB construct the modulus figure for the compression area above AB. Now consider an elementary strip PQ of thickness δy at a distance y from the N.A.

The area of the strip $\delta a = PQ.\delta y$.

The intensity of stress on this area is given by,

$\frac{p}{y} = \frac{f_c}{h}$ or $p = f_c \times \frac{y}{h}$.

The elementary thrust δC on the area δa is therefore, $\delta C = p.\delta a = f_c \times \frac{y}{h} \times \delta a = f_c \times \delta a_1$, where δa_1 is the area of the corresponding strip in the modulus figure. The resultant thrust on the compression area is,

$C = \Sigma\, \delta C = \Sigma f_c.\delta a_1 = f_c.a_1$.

The effect of the *varying* bending stress on the compression area is thus seen to be equivalent to a *uniform* stress f_c over the corresponding area of the modulus figure. We have thus reduced the case of a varying stress on a given area to one of uniform stress on a modified area — the modulus figure — which is naturally less than the given area.

Applying this to a rectangular section of breadth b and depth d, construct the modulus figures for the compression and tension areas with the pole O at the c.g. of the section (see fig. 93).

The modulus figures are triangles whose areas a_1 are $\frac{1}{2}b.\frac{d}{2} = \frac{bd}{4}$. If f is the extreme intensity of stress in tension or compression, $C = T = f.a_1 = \frac{f.b.d}{4}$.

Since the intensity of stress f is uniform over the modulus figure, C and T will act at the centres of gravity of their modulus figures i.e. at $\frac{2}{3} \times \frac{d}{2}$ or $\frac{d}{3}$ from O. The lever arm of the moment of resistance formed by C and T is therefore $\frac{2d}{3}$ and the moment of resistance is,

$$M_r = C \text{ or } T \times \text{arm} = \frac{f.b.d}{4} \times \frac{2d}{3} = \frac{1}{6} f.b.d^2 \text{ as already}$$

obtained.

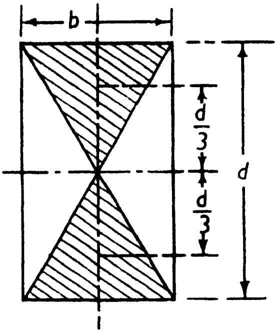


FIG. 93

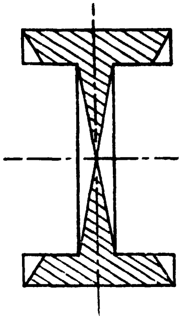


FIG. 94

The modulus figures for the compression and tension areas of an *I*-section are shown in fig. 94.

15. Beams of uniform strength: Usually beams have the same sectional area throughout the span but as mentioned earlier, it is wasteful to have a uniform section throughout for beams of large spans carrying heavy loads. In such cases, it is customary to design a section to resist the B.M. on it at the maximum permissible skin stress. Such beams of variable section are known as beams of "Uniform Strength" to distinguish them from beams of "Uniform Section". The section may be varied by (i) keeping uniform width throughout and varying the depth, (ii) by keeping constant depth and varying the breadth or (iii) by altering both the depth and the width. Plate girders of constant depth and curtailed flange-plates, plate and lattice girders with inclined chords are usually beams of uniform strength.

For an elementary illustration of a beam of uniform strength, consider a beam of rectangular section carrying a point load W at the centre of span l. The bending moment at a distance

$x(<\frac{l}{2})$ from either support is $\frac{W}{2}.x$. The resisting section may be designed with a uniform width b throughout and a variable depth d_x which will be obtained by equating the bending moment to the moment of resistance at the maximum permissible skin stress.

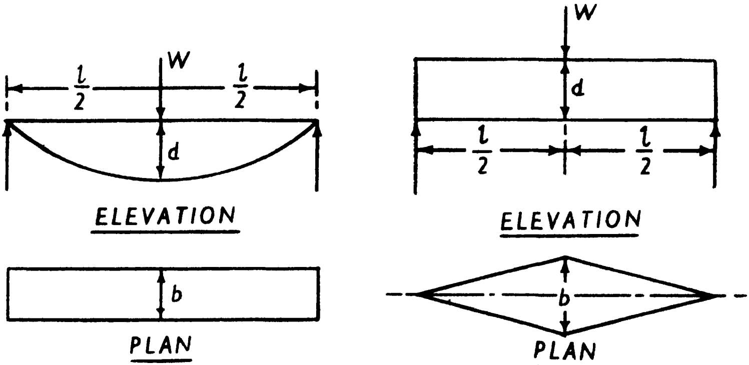
Thus,

$$f \times \tfrac{1}{6} \times b \times d_x^2 = \frac{W}{2}.x$$

$$\therefore \qquad d_x^2 = \frac{3W.x}{f.b}$$

$$\text{or} \qquad d_x = \sqrt{\frac{3W.x}{f.b.}}.$$

The depth d_x is proportional to $\sqrt{x}$. At the centre, the depth d will be $\sqrt{\frac{3Wl}{2fb}}$. The plan and elevation of such a beam are shown in fig. 95(a).



(a) (b)

FIG. 95

If the beam is designed with a variable width b_x and a constant depth d, b_x is given by,

$$f \times \tfrac{1}{6} \times b_x \times d^2 = \frac{W}{2}.x.$$

$$\therefore \quad b_x = \frac{3W}{f.d^2}.x.$$

The width is thus proportional to x. At the centre its value $b = \frac{3W.l}{2f.d^2}$.

The plan and elevation of a beam of constant depth and variable width are shown in fig. 95(*b*).

It should be needless to say that the section discussed above is only intended to indicate how the width or depth will vary for a beam of uniform strength, of rectangular section with a central load.

16. Resistance to Shear Force: Shear Stresses: At any cross-section of a beam, the two actions to be resisted are the Bending Moment and the Shear Force. On certain simple assumptions we have seen how the section resists the bending moment. We shall now investigate the nature of resistance set up by the section to the Shear Force.

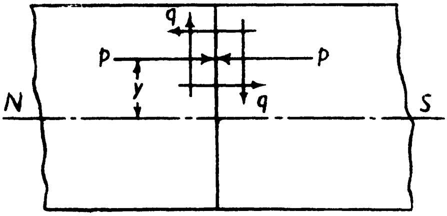


FIG. 96

Let M and F be the bending moment and shear force at a given section. Let p be the intensity of stress at a given point in the section to resist M. The stress p is compressive or tensile depending upon its position with regard to the neutral axis. Let q be the intensity of shear stress at the point to resist the shear force F. We have seen that a shear stress across a given plane cannot exist unless it is accompanied by a shear stress of the same intensity across a plane at right angles. The shear stress q at the point acting across the vertical section must therefore be accompanied by a horizontal shear stress of intensity about the N.A. and $\sum_{y_1}^{y_C} y.\delta a$ is the sum of the moments of all the elementary areas from y_1 to y_C about the N.A. But this is equal to the moment of the beam section area from y_1 to y_C about the N.A. We therefore deduce that,

$q_1 = \frac{F}{I.z_1}$ × moment of the sectional area from y_1 to y_C about the N.A.

This is the intensity of horizontal shear stress at a height y_1 from the N.A. But the intensity of vertical shear must be the same. Dropping the suffixes we get,

$q = \frac{F}{I.z}$ × moment, about N.A., of the sectional area from top to the point under study where z is the width of section at the point and I is the moment of Inertia of the beam section about the N.A.

17. Shear stresses in beams of rectangular and circular sections: (i) Rectangular section: width b, depth d. To evaluate q at a distance y from the N.A.

In fig. 98, the shaded area $= b\ (\frac{d}{2} - y)$.

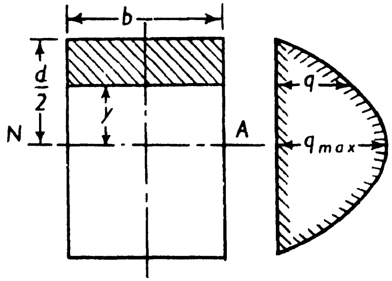


FIG. 98

The distance of its c.g. from the N.A. is the mean of y and $\frac{d}{2}$ i.e. $\frac{1}{2}\ (\frac{d}{2} + y)$. The moment of the area about the N.A.

$$= b\ (\frac{d}{2} - y) \times \frac{1}{2}\ (\frac{d}{2} + y) = \frac{b}{2}\ (\frac{d^2}{4} - y^2).$$

The intensity of shear stress is,

$$q = \frac{F}{I.b} \times \frac{b}{2}\left(\frac{d^2}{4} - y^2\right) = \frac{F}{2I}\left(\frac{d^2}{4} - y^2\right)$$

q increases as y diminishes until at the N.A. where y is zero, q is a maximum. The variation of q with regard to y is parabolic, whereas the bending stress p varies by a straight line law.

At the top, the value of q is zero. At the N.A.,

$$q_{max} = \frac{Fd^2}{8I}.$$

Since $I = \frac{1}{12}bd^3$, $q_{max} = \frac{12F.d^2}{8 \cdot bd^3} = \frac{3F}{2bd} = 1 \cdot 5 q_{average}.$

The maximum intensity of shear stress for a rectangular section is 50% higher than the average. Below the N.A. the value of q diminishes, becoming zero at the bottom. See fig. 98 for the stress distribution.

It is interesting to note how Nature compensates. Of the two actions M and F at a section, we have seen that where the B.M. is a maximum, the S.F. is usually zero. Where neither is zero while the effects of M are maximum at the extremities and zero at the N.A., the effects of F are just the reverse, being zero at the extremities and maximum at the N.A.

(ii) Solid circular section: diameter d.

At a height y above the N.A., width of section $z = 2\sqrt{r^2 - y^2}$ where r is the radius of the section.

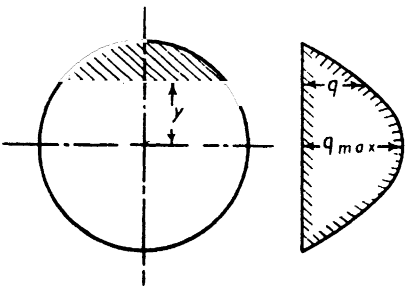


FIG. 99

Elementary area of a strip of thickness δy is.
$\delta a = 2\sqrt{r^2 - y^2}.\ \delta y.$

The moment of δa about the N.A.

$$= y.\delta a = 2y\sqrt{r^2-y^2}.\delta y.$$

The moment of the shaded area about the N.A.

$$= \int_y^r 2y\sqrt{r^2-y^2}.\delta y = \int_y^r y.z.dy.$$

Since $z = 2\sqrt{r^2-y^2}$

$$z^2 = 4\ (r^2-y^2)$$

$\therefore$ $2z.\delta z = -\,8y.\delta y$

or $y\delta y = -\,\frac{1}{4}z.\delta z.$

Substituting, the moment of the shaded area

$$= \int_z^0 -\frac{z^2}{4}\,dz = \frac{1}{4}\int_0^z z^2dz = \frac{z^3}{12}$$

$$q = \frac{F}{I.z} \times \text{moment of shaded area}$$

$$= \frac{F}{I.z} \times \frac{z^3}{12} = \frac{F}{12\times I}.z^2 = \frac{F}{12\times I} \times 4(r^2-y^2) = \frac{F}{3I}(r^2-y^2).$$

The intensity of shear stress q again varies with regard to y by a parabolic law, being zero at the extremities and reaching a maximum of $\frac{F.r^2}{3\times I}$ or $\frac{F.d^2}{12\times I}$ at the N.A.

I for a solid circular section is $\frac{\pi d^4}{64}$

$$\therefore\quad q_{max} = \frac{64}{12}.\frac{F.d^2}{\pi d^4} = \frac{16}{3}.\frac{F}{\pi d^2} = \frac{4}{3}.\frac{F}{\text{area}} = \frac{4}{3}.\,q_{average} = 1{\cdot}33q_{ave.}$$

For a solid circular section the maximum intensity is $33\frac{1}{3}\%$ higher than the average.

18. Shear stresses in beams of I-section: Let B and D be the overall width and depth of section and b and d the thickness and depth of web.

In the flange:

At a distance y from the N.A., width of section $= B$.

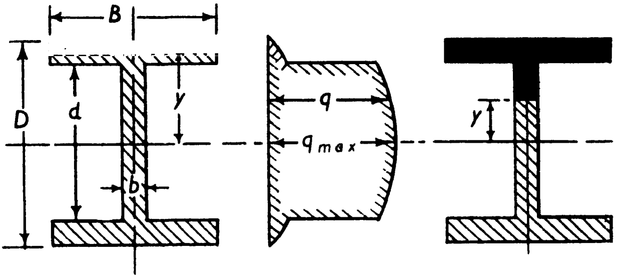
Shaded area fig. 100(a) $= B\left(\frac{D}{2} - y\right)$, the distance of its c.g. from the N.A. $= \frac{1}{2}\left(\frac{D}{2} + y\right)$

$$\therefore \quad q = \frac{F}{I.B} \times B\left(\frac{D}{2} - y\right) \times \frac{1}{2}\left(\frac{D}{2} + y\right)$$

$$= \frac{F}{2I}\left(\frac{D^2}{4} - y^2\right), \text{ which is parabolic.}$$

At the junction with the web,

$y = \frac{d}{2}$ and q reaches the value $\frac{F}{8I}(D^2 - d^2)$



(a) (b) (c)

Fig. 100

In the web:

At junction with the flange, the width suddenly changes from B to b. The value of q therefore suddenly changes from $\frac{F}{8I}(D^2 - d^2)$ to $\frac{B}{b} \times \frac{F}{8I}(D^2 - d^2)$.

At a distance y from the N.A., the width of section is b. The shaded area fig. 100(c) can be split up into two rectangles consisting of the flange and web portions. Its moment about the N.A. will be the sum of the moments of the two rectangles i.e.,

$$B\left(\frac{D-d}{2}\right) \times \frac{1}{2}\left(\frac{D}{2} + \frac{d}{2}\right) + b\left(\frac{d}{2} - y\right) \times \frac{1}{2}\left(\frac{d}{2} + y\right)$$

or $$\frac{B(D^2 - d^2)}{8} + \frac{b}{2}\left(\frac{d^2}{4} - y^2\right).$$

$$\therefore\quad q = \frac{F}{I.b}\left\{\frac{B\,(D^2-d^2)}{8} + \frac{b}{2}\,(\frac{d^2}{4} - y^2)\right\}$$

$$= \frac{B}{b} \times \frac{F}{8I}(D^2-d^2) + \frac{F}{2I}\,(\frac{d^2}{4} - y^2),$$ which is also parabolic.

The maximum intensity occurs at the N.A. where $y = 0$.

$$\therefore\quad q_{max} = \frac{B}{b} \times \frac{F}{8I}(D^2-d^2) + \frac{F.d^2}{8I}.$$

The stress distribution is shown in fig. 100(*b*). It will be noted that most of the shear is resisted by the web, the resistance offered by the flanges being negligible. The practical utility of the *I*-section will now be evident. Of the two actions on a section, the flanges resist most of the bending moment, while the web resists most of the shear force.

Example 22.

A beam of I-section is 10″ deep by 8″ wide. The flanges are 1″ thick while the web thickness is ·6″. At a cross-section, the bending moment is M and the force is F. Calculate the proportion in which the flanges and the web resist (i) M and (ii) F.

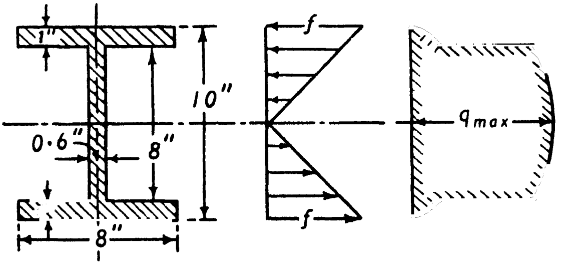


FIG. 101

The moment of Inertia of the section about its N.A.

$$= \frac{1}{12}\,(8 \times 10^3 - 7{\cdot}4 \times 8^3) = \frac{1}{12}\,(8000 - 3788{\cdot}8) = 351 \text{ in}^4.$$

(i) Resistance to *M*:

Consider an elementary strip δa in the web, of thickness δy at a distance y from the N.A. where the bending stress p will be

$\frac{M.y}{I}$. The elementary thrust δC on the strip is $p.\delta a = \frac{M.y}{I} \times \cdot 6\delta y$. Its moment about the N.A. $= \frac{M.y}{I} \times \cdot 6 \times \delta y \times y$
$= \frac{\cdot 6M}{I}.y^2.\delta y.$

The moment of resistance offered by the web is,

$$M_w = \int_{-4}^{+4} \frac{\cdot 6M}{I}.y^2dy = \frac{1\cdot 2M}{I}\int_0^4 y^2dy = \frac{1\cdot 2M}{I} \times \frac{64}{3} = \frac{25\cdot 6M}{I}$$

$$= \frac{25\cdot 6}{351} \times M = \cdot 073M.$$

The moment of resistance offered by the flanges is the balance viz., $\cdot 927M$. The flanges take up 92·7% of the B.M., the web resisting only 7·3%.

(ii) Resistance to F:

Consider an elementary area δa in the flange, of thickness δy at a distance y from the N.A.

Then $\delta a = 8\cdot\delta y$.

The shear intensity $q = \frac{F}{I \times 8}\ [8(5-y) \times \frac{1}{2}(5+y)]$

$$= \frac{F}{2I}(25 - y^2).$$

The shear borne by the elementary strip δa

$$= q.\delta a = \frac{F}{2I}(25-y^2) \times 8\delta y = \frac{4F}{I}(25 --y^2)\ \delta y.$$

The shear resistance offered by one flange

$$= \int_4^5 \frac{4F}{I}(25-y^2)\ dy = \frac{4F}{I}\left\{25y - \frac{y^3}{3}\right\}_4^5$$

$$= \frac{4F}{I}[25(5-4) - \frac{1}{3}(125 \quad 64)] = \frac{56F}{3I}.$$

The shear resistance offered by both the flanges

$$= \frac{112F}{3I} = \frac{112F}{3 \times 351} = \cdot 1063F.$$

The shear borne by the web is the balance viz., $\cdot 8937F$. The web thus takes up 89·37% of the shear force, the flanges resisting only 10·63%.

In the design of built-up I-sections as for plate girders, the work is simplified by making the flanges strong enough to take up the entire bending moment and the web to resist the whole of the shear force at the section.

Example 23.

A 14″ × 6″ R.S.J. has flanges ·7″ thick while the web thickness is ·4″. At a certain cross-section the shear force is 20 tons. Calculate the maximum intensity of shear stress and sketch the distribution of shear stress across the section.

$I = \frac{1}{12}(6 \times 14^3 - 5{\cdot}6 \times 12{\cdot}6^3) = 438 \text{ in}^4.$

The intensity of shear stress in the flange at junction with the web is,

$$q = \frac{F}{IB} \times B\left(\frac{D}{2} - \frac{d}{2}\right) \times \tfrac{1}{2}\left(\frac{D}{2} + \frac{d}{2}\right)$$

$$= \frac{F}{I} \times {\cdot}7 \times \tfrac{1}{4}(7 + 6{\cdot}3) = \frac{{\cdot}7 \times 6{\cdot}65 \times 20}{438} = {\cdot}2125 \text{ ton/in}^2.$$

This will suddenly increase $\frac{B}{b}$ times at a point in the web at the junction with the flange. This value is $\frac{6}{\cdot 4}$ or 15 times ·2125 i.e. 3·1875 tons/in².

The maximum intensity occurs at the N.A.

$$q_{max} = \frac{20}{438 \times {\cdot}4}\left[6 \times {\cdot}7 \times \tfrac{1}{2}(7 + 6{\cdot}3) + {\cdot}4 \times 6{\cdot}3 \times \frac{6{\cdot}3}{2}\right]$$

$$= 3{\cdot}1875 + \frac{20}{438} \times \frac{6{\cdot}3^2}{2} = 3{\cdot}1875 + {\cdot}9059$$

$$= 4{\cdot}09 \text{ tons/in}^2.$$

For a sketch of the distribution of stress, refer to fig. 100(b).

19. Shear stresses in angle and Tee-sections:

Example 24.

The cross-section of a joist is a Tee-section 4″ × 6½″ × ½″ with *the 4" side horizontal. Find the maximum intensity of shear stress and sketch the distribution of stress across the section if it has to resist a shear force of 10 tons.*

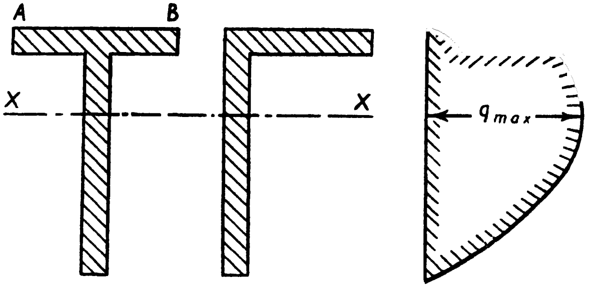


FIG. 102

Area of flange $a_1 = 4 \times \frac{1}{2} = 2$ in^2.; $y_1 = \frac{1}{4}''$, y_1 being the distance of its c.g. from *AB*.

$$\text{Area of web } a_2 = 6 \times \tfrac{1}{2} = 3 \text{ in}^2.;\ y_2 = \tfrac{1}{2} + 3 = 3\tfrac{1}{2}''$$

$$\bar{y} = \frac{a_1y_1 + a_2y_2}{a_1 + a_2} = \frac{2 \times \frac{1}{4} + 3 \times 3{\cdot}5}{5} = 2{\cdot}2''$$

$$\begin{aligned} I_{AB} &= \tfrac{1}{3} \times 4 \times (\tfrac{1}{2})^3 + \tfrac{1}{12} \times \tfrac{1}{2} \times 6^3 + 3 \times (\tfrac{7}{2})^2 \\ &= 45{\cdot}92 \text{ in}^4. \\ &= I_{xx} + 5 \times (2{\cdot}2)^2 \\ \therefore \quad I_{xx} &= 45{\cdot}92 - 24{\cdot}20 = 21{\cdot}72 \text{ in}^4. \end{aligned}$$

The intensity of shear stress in the flange at junction with the web is,

$$q = \frac{F}{I \times 4} \times 4 \times \tfrac{1}{2} \times (2{\cdot}2 \quad \tfrac{1}{4}) = \frac{10 \times 2 \times 1{\cdot}95}{4 \times 21{\cdot}72} = {\cdot}449 \text{ ton/in}^2.$$

In the web at junction with the flange, the value suddenly goes up to $\frac{4}{{\cdot}5} \times {\cdot}449 = 3{\cdot}59$ tons/in^2.

The maximum value at the N.A. is,

$$\begin{aligned} q_{max} &= \frac{F}{I \times {\cdot}5}\left\{4 \times \tfrac{1}{2} \times (2{\cdot}2 - {\cdot}25) + {\cdot}5 \times 1{\cdot}7 \times \frac{1{\cdot}7}{2}\right\} \\ &= \frac{20}{21{\cdot}72}[3{\cdot}90 + {\cdot}723] = \frac{20 \times 4{\cdot}623}{21{\cdot}72} = 4{\cdot}26 \text{ tons/in}^2. \end{aligned}$$

20. Shear stresses in built-up sections: These are indicated in fig. 103.

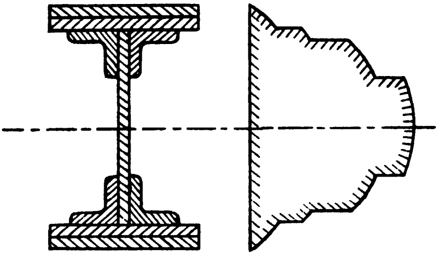


FIG. 103

21. Beam of square section with one diagonal horizontal: As an interesting exercise, we shall now discuss the shear intensity at a point in a square section with a horizontal diagonal of length b. The moment of Inertia of the section about the diagonal AB

$$= 2 \times \frac{1}{12} \times b \times \left(\frac{b}{2}\right)^3 = \frac{b^4}{48}.$$

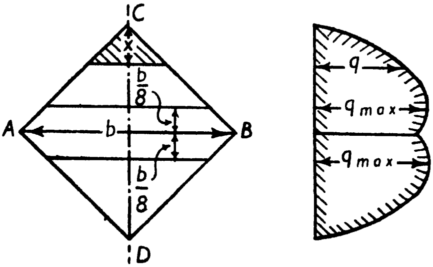


FIG. 104

Consider a point in the section distant x from C.

The shaded area $= \frac{1}{2} \times 2x \times x = x^2$.

The distance of its c.g. from the N.A.

$$= \left(\frac{b}{2} - \frac{2x}{3}\right) = \frac{(3b - 4x)}{6}.$$

The intensity of shear stress at the point is given by,

$$q = \frac{F}{I \times 2x} \times x^2 \times \frac{(3b - 4x)}{6} = \frac{48F.x^2(3b - 4x)}{12x.b^4}$$

$$= \frac{4F.x\ (3b - 4x)}{b^4}.$$

The variation of the stress intensity with regard to x is parabolic.

For a maximum, differentiate the expression with regard to x and equate to zero.

$$\therefore \quad 3b - 8x = 0 \quad \text{i.e.} \quad x = \frac{3b}{8}.$$

The maximum occurs at $\frac{3b}{8}$ from C or at $\frac{b}{8}$ from AB.

$$q_{max} = \frac{4F}{b^4} \times \frac{3b}{8} \times \left(3b - \frac{4 \times 3}{8}b\right) = \frac{9F}{4b^2}.$$

The area of the square section

$$= 2 \times \frac{b}{2} \times \frac{b}{2} = \frac{b^2}{2} \quad \therefore \quad q_{ave} = \frac{2F}{b^2} \text{ and } q_{max} = \frac{9}{8} \times q_{ave}.$$

At the N.A., $x = \frac{b}{2}$ and the value of q is

$$q_0 = \frac{4F}{b^4} \times \frac{b}{2} \times \left(3b - 4 \times \frac{b}{2}\right) = \frac{2F}{b^2} = q_{ave}.$$

At the N.A., the actual shear intensity is equal to the average intensity.

The distribution of stress intensity across the section is shown in fig. 104.

22. Principal stresses and Principal planes at a point in a beam section: In article 16 of this chapter it has been mentioned that the nature of stresses set up at a point in a beam section is a direct stress p accompanied by a state of simple shear q. At a point in the beam section, distant y from the N.A., the bending stress will be $p = \frac{M.y}{I}$; the shear stress will be $q = \frac{F}{I.z}\sum_{y}^{y_c} y.\delta a$. The position of the principal planes through the point is given by $\tan 2\theta = \frac{2q}{p}$ where θ is the inclination of a principal plane to the plane of p.

The principal stresses are:

$$p_1 = \frac{p}{2} + \sqrt{\frac{p^2}{4} + q^2}$$

and $$p_2 = \frac{p}{2} - \sqrt{\frac{p^2}{4} + q^2}.$$

The major principal stress will be of the same sign as p i.e. it will be compressive on one side of the N.A. and tensile on the other. The second principal stress p_2 is opposite in sign to p_1.

Example 25.

At a certain cross-section of a beam of rectangular section 5″ wide × 12″ deep, the bending moment is 192000 lb-inches while the shear force is 4800 lb-wt. Calculate the bending and shear stress at 6″, 4″, 2″ and 0″ from the neutral axis. Find the principal planes and principal stresses at these points.

I for the section $= \frac{1}{12} \times 5 \times 12^3 = 720$ in^4.

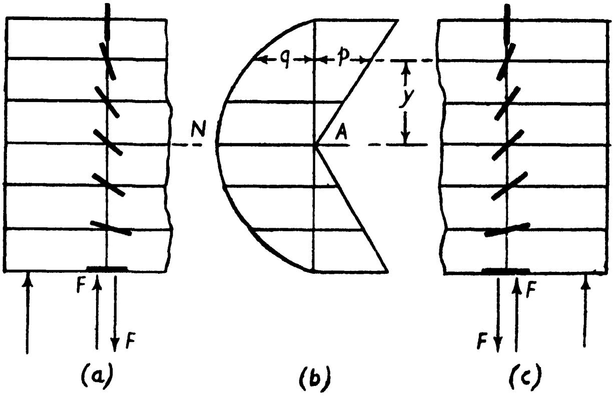


FIG. 105

At 6″ above the N.A.,

$$p = \frac{M.y}{I} = \frac{192000 \times 6}{720} = 1600 \text{ lb/in}^2\text{., compressive}$$

$$q = 0.$$

The major principal stress $p_1 = p = 1600$ lb/in^2., compressive, and $p_2 = 0$.

At 4″ above the N.A.,

$$p=\frac{192000\times 4}{720}=\frac{3200}{3}\ \text{lb/in}^2\text{., compressive.}$$

$$q=\frac{4800}{720\times 5}\times 5\ (6-4)\times 5=\frac{200}{3}\ \text{lb/in}^2.$$

For the position of the principal planes,

$$\tan 2\theta=\frac{2q}{p}=\frac{2\times 200}{3200}=\cdot 125$$

$$\therefore\quad 2\theta=7°7'$$

$$\therefore\quad \theta_1=3°34'$$

$$\text{and}\quad \theta_2=93°34'$$

$$p_1=\frac{p}{2}+\sqrt{\frac{p^2}{4}+q^2}=\frac{1600}{3}+\sqrt{\left(\frac{1600}{3}\right)^2+\left(\frac{200}{3}\right)^2}$$

$$=\frac{1600}{3}+\frac{1612\cdot 4}{3}=1070\cdot 8\ \text{lb/in}^2\text{., compressive.}$$

$$p_2=\frac{p}{2}-\sqrt{\frac{p^2}{4}+q^2}=\frac{1600}{3}-\frac{1612\cdot 4}{3}$$

$$=-4\cdot 13\ \text{or}\ 4\cdot 13\ \text{lb/in}^2\text{., tensile.}$$

At 2″ above the N.A.,

$$p=\frac{192000\times 2}{720}=\frac{1600}{3}\ \text{lb/in}^2\text{., compressive.}$$

$$q=\frac{4800}{720\times 5}\times 5\ (6-2)\times 4=\frac{320}{3}\ \text{lb/in}^2.$$

$$\therefore\quad \tan 2\theta=\frac{2q}{p}=\frac{640}{1600}=\cdot 40$$

$$\therefore\quad 2\theta=21°48'$$

$$\therefore\quad \theta_1=10°54'$$

$$\text{and}\quad \theta_2=100°54'$$

$$p_1=\frac{p}{2}+\sqrt{\frac{p^2}{4}+q^2}=\frac{800}{3}+\sqrt{\left(\frac{800}{3}\right)^2+\left(\frac{320}{3}\right)^2}$$

$$=\frac{800}{3}+\frac{861\cdot 6}{3}=553\cdot 86\ \text{lb/in}^2\text{., compressive.}$$

$$p_2=\frac{800}{3}-\frac{861\cdot 6}{3}=-20\cdot 53\ \text{lb/in}^2\text{.,}$$

or 20·53 lb/in²., tensile.

At the N.A.,

$$p = 0$$

$$q = \frac{4800}{720 \times 5} \times 5 \times 6 \times 3 = 120 \text{ lb/in}^2.$$

At the N.A., there is just a state of simple shear of intensity 120 lb/in^2. The diagonal planes are principal planes.

$$p_1 = +q = 120 \text{ lb/in}^2\text{., compressive.}$$

$$p_2 = -q = 120 \text{ lb/in}^2\text{., tensile.}$$

Below the N.A., the values of the principal stresses are numerically the same as for corresponding points above the N.A. but opposite in sign.

The position of the principal planes is marked in figs. 105(*a*) and 105(*c*). In fig. 105(*a*), the nature of the shear force is ↑ ↓ while in fig. 105(*c*) it is ↓ ↑. The planes carrying compressive stresses are marked in thick lines. It is instructive to follow up the inclinations of one set of principal planes at a cross-section, say the compression ones, from top to bottom. At the top the compression plane is vertical; as we proceed lower, its inclination to the vertical increases until at the neutral axis the inclination is 45°. As we proceed below the N.A., the inclination continues to increase until at the bottom the plane is horizontal. The principal compressive stress is maximum at the top where its plane is vertical. Proceeding lower it diminishes until at the bottom its intensity is zero while its plane is horizontal.

23. Curves of Principal Stresses: As indicated in the preceding example, the position of the principal planes at all points in a cross-section top to bottom can be marked out. This can be done for a number of sections from one end of the beam to the other. It is then possible to draw curves which will be the "Envelopes" of the principal planes i.e. which at every point are tangential to some principal plane. There will be two sets of these curves which may be called "Curves of Principal Stress". The following properties of the two sets of curves of principal stresses will be evident from a study of fig. 106.

(i) They are such that the tangent and normal at any point on a curve give the directions of the two principal stresses at the point.

(ii) They cross each other at right angles.

(iii) They cross the neutral axis at 45° to it.

(iv) They cut the top and bottom surfaces at 90°.

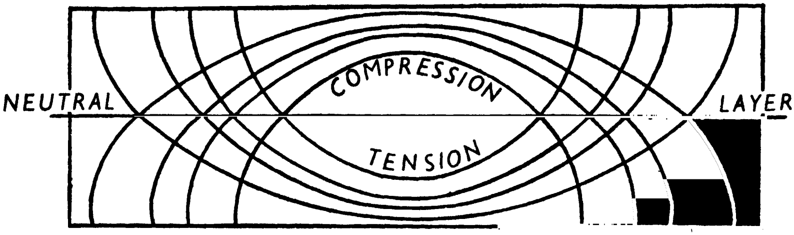


FIG. 106

(v) The intensity of stress along any one curve is greatest when it is parallel to the length of the beam, diminishing along the curve until it is zero where it meets the top or bottom surface of the beam at right angles.

24. Principal Stresses in an I-section: Ordinarily the formula $M = fZ$ is used in the design of a beam section for selecting a suitable section-modulus Z to resist the bending moment M so that the skin stress does not exceed a given working stress f for the material. This assumes that the skin stress which is also a principal stress, is the maximum that occurs anywhere in the cross-section and that the section is safe if this does not exceed the permissible working stress. For the *I*-section however, a point in the web at junction with the flange sometimes needs watching. The bending stress p at the point though less than the skin stress is high. The shear intensity q, likewise, is also high though it is less than the maximum intensity at the N.A. So when the principal stresses for the point are worked out, the major principal stress p_1 which is $\left\{\frac{p}{2} + \sqrt{\frac{p^2}{4} + q^2}\right\}$ may even exceed the skin stress as will be clear from the following example.

Example 26.

A 10″ × 8″ × 70″ lb. R.S.J. has flanges 1″ thick while the web thickness is ·6″. At a certain cross-section it has to resist a bending moment of 500 ton-inches and a shear force of 25 tons. Find the principal planes and principal stresses (a) at the top (b) in the web at 4″ from the N.A. (c) at the N.A.

The moment of Inertia of the section about the N.A. = $\frac{1}{12}\{8 \times 10^3 - 7{\cdot}4 \times 8^3\} = 351$ in^4.

(a) At the top:

$$p = \frac{500 \times 5}{351} = 7{\cdot}12 \text{ tons/in}^2\text{., compressive.}$$

$$q = 0.$$

The major principal stress $p_1 = 7{\cdot}12$ tons/in^2., compressive.

The minor principal stress $p_2 = 0$.

(b) In the web, 4″ from the N.A.:

$$p = \frac{M.y}{I} = \frac{500 \times 4}{351} = 5{\cdot}7 \text{ tons/in}^2\text{., compressive.}$$

$$q = \frac{25}{351 \times {\cdot}6} \times 8 \times 1 \times 4\tfrac{1}{2} = 4{\cdot}27 \text{ tons/in}^2.$$

For the position of the principal planes,

$$\tan 2\theta = \frac{2q}{p} = \frac{8{\cdot}54}{5{\cdot}7} = 1{\cdot}498$$

$$\therefore \quad 2\theta = 56°18'$$

$$\therefore \quad \theta_1 = 28°9'$$

and $\theta_2 = 118°9'$.

The principal stresses are,

$$p_1 = \frac{p}{2} + \sqrt{\frac{p^2}{4} + q^2} = 2{\cdot}85 + \sqrt{(2{\cdot}85)^2 + (4{\cdot}27)^2}$$

$$= 2{\cdot}85 + 5{\cdot}13 = 7{\cdot}98 \text{ tons/in}^2\text{., compressive.}$$

$$p_2 = \frac{p}{2} - \sqrt{\frac{p^2}{4} + q^2} = 2{\cdot}85 - 5{\cdot}13 = 2{\cdot}28 \text{ tons/in}^2\text{., tensile.}$$

It is to be noted that the major principal stress p_1 at this point is nearly 8 tons/in². and exceeds the skin stress of 7·12 tons/in².

(c) At the N.A.:

$$p = 0$$

$$q = \frac{25}{351 \times \cdot 6}\left\{8 \times 1 \times \frac{9}{2} + \cdot 6 \times 4 \times 2\right\} = 4{\cdot}27 + \cdot 57$$

$$= 4{\cdot}84 \text{ tons/in}^2.$$

The principal stresses across the diagonal planes are 4·84 tons/in²., compressive on one plane and 4·84 tons/in²., tensile on the other. The variation of principal stresses across the section is shown in fig. 107.

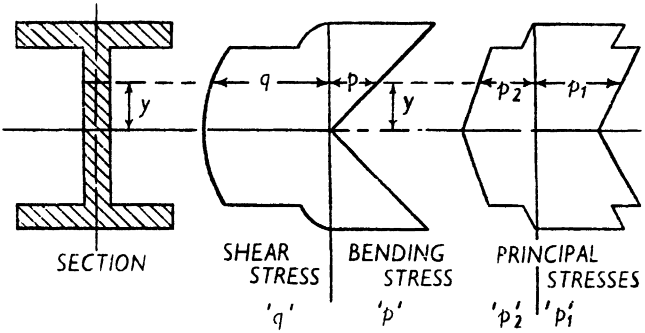


FIG. 107

It is only in exceptional cases of heavy loading that the skin stress is exceeded by the major principal stress in the web at junction with the flange. For ordinary designs it is not necessary even to check up shear effects. The section is designed for the maximum bending moment. The shear force at such a section is usually zero, in which case, there are no shear stresses at the section. If the beam is given a uniform section throughout, the section provided is generally ample for resisting the maximum shear that may come anywhere in the beam as will be seen in the following example.

Example 27.

A load of 10 tons is uniformly spread over a beam freely supported over a span of 16 feet. Select a suitable R.S.J. if the maximum bending *stress is limited to $7\frac{1}{2}$ tons/in^2. Calculate the maximum intensity of shear stress anywhere in the section so selected.*

$$\text{The maximum B.M.} = \frac{W.l}{8} \times 12 \text{ ton-inches.}$$

$$= \frac{10 \times 16 \times 12}{8} = 240 \text{ ton-inches.}$$

$$\text{Section modulus required} = \frac{M_r}{f} = \frac{240}{7{\cdot}5} = 32 \text{ in}^3.$$

Select a 12″ × 5″ × 32 lb. R.S.J. which has a section modulus of 36·66 in^3.

The flange and web thicknesses of this section are ·55″ and ·35″ respectively. $I_{xx} = 220$ in^4.

The maximum shear is 5 tons at each support.

The maximum intensity of stress due to this shear will be at the N.A. and is,

$$q_{max} = \frac{5}{220 \times {\cdot}35}\left\{5 \times {\cdot}55 \times 5{\cdot}725 + {\cdot}35 \times 5{\cdot}45 \times \frac{5{\cdot}45}{2}\right\}$$

$$= 1{\cdot}022 + {\cdot}338 = 1{\cdot}36 \text{ tons/in}^2.$$

The maximum intensity of shear stress is only 1·36 tons/in^2. against the permissible stress of 5 to 6 tons/in^2. The selected section is thus ample to resist the maximum shear at the supports.

25. Beams of composite section; flitched beams: A flitched beam is a wooden beam reinforced by a mild steel plate. It consists of two wooden joists with a steel plate sandwiched between them, all the three being bolted together so that they behave as one composite beam section. Let B be the combined width and D the depth of the joists; let t be the thickness and d the depth of the steel plate. If f_w is the intensity of skin stress in the wooden joists, their moment of resistance $M_w = f_w \times \frac{1}{6}B.D^2$.

The corresponding skin stress in the steel plate can be deduced as follows:

Since the joists and the plate are firmly held together, the deformation due to bending in the joists or the plate at a given distance from the N.A. is the same.

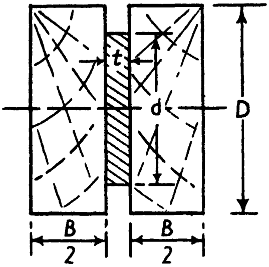


FIG. 108

At $\frac{d}{2}$ from the N.A., the bending stress in the wooden joists is given by,

$$\frac{p_w}{d/2} = \frac{f_w}{D/2} \text{ or } p_w = f_w \times \frac{d}{D}.$$

The strain is therefore given by,

$e = \frac{p_w}{E_w}$ where E_w is the modulus of Elasticity for wood.

The corresponding stress in the steel plate at this level is given by

$\frac{f_t}{E_s} = e$ or $f_t = E_s.e$ where E_s is Young's modulus for steel, being the common strain at this level.

$\therefore \; f_t = E_s \times \frac{p_w}{E_w} = \frac{E_s}{E_w} \times \frac{d}{D} \times f_w = m \times \frac{d}{D} \times f_u$, using m for the modular ratio of steel and wood.

The moment of resistance of the steel plate is therefore,

$M_s = f_t \times \frac{1}{6} \times t \times d^2$ where $f_t = m \times \frac{d}{D} \times f_w$.

The total moment of resistance of the composite section is given by,

$$M_r = M_w + M_s = f_w \times \frac{1}{6} BD^2 + f_t \times \frac{1}{6} t.d^2.$$

Example 28.

A flitched timber consists of two joists $4\frac{1}{2}''$ wide and $12''$ deep with a steel plate $10''$ deep and $\frac{1}{2}''$ thick placed symmetrically between and clamped to them. When the fibre stress in the joists reaches 1000 lb/in^2., what is the corresponding skin stress in the steel plate and what is the total moment of resistance of the composite section? Take $E_s = 18E_w$.

$$f_t = m \times \frac{d}{D} \times f_w = 18 \times \tfrac{10}{12} \times 1000 = 15000 \text{ lb/in}^2.$$

$$\begin{aligned} M_w &= 1000 \times \tfrac{1}{6} \times 9 \times 12 \times 12 = 216000 \text{ lb-inches.} \\ M_s &= 15000 \times \tfrac{1}{6} \times \tfrac{1}{2} \times 10 \times 10 = 125000 \text{ lb-inches.} \\ &\text{Total moment of resistance} = 216000 + 125000 \\ &\phantom{\text{Total moment of resistance}} = 341000 \text{ lb-inches.} \end{aligned}$$

A concrete beam reinforced with steel is another example of a beam of composite section. Reinforced concrete beams and columns will be studied separately in subsequent chapters.

EXAMPLES V

(1) A beam resting freely on supports 16 feet apart, carries a uniformly distributed load of 750 lb. per foot run and also a point-load of 2000 lb-wt. at 6 feet from the left support. If the permissible stress in timber is 1200 lb/in^2., design a suitable section by making the depth twice the width. ($M_{\max} = 30375$ lb-feet; $8'' \times 16''$.)

(2) A rectangular beam $6''$ wide and $12''$ deep is freely supported over a span of 16 feet and carries a load of 200 lb. per foot run. It also carries three equal point loads W lb. each, equispaced over the beam. If the permissible bending stress is 1000 lb/in., find the max. allowable value of W. (700 lb-wt.)

(3) A floor load of 150 lb. per square foot is carried on rolled steel joists spaced 4 feet apart centre to centre. If the span is 16 feet and the maximum permissible stress is $7\frac{1}{2}$ ton/in^2., calculate the section modulus necessary for each joist and suggest a suitable section. ($Z = 13{\cdot}72$ in^3.; $8'' \times 4'' \times 18$ lb.)

(4) A water main 4 feet internal diameter is made of mild steel plate $\frac{1}{2}''$ thick and is running full. If it is freely supported at ends, find the maximum permissible span if the bending stress is not to exceed 8000 lb/in^2. Weight of steel $= 480$ lb/ft^3.; weight of water $= 62{\cdot}5$ lb/ft^3. (68′ 6″.)

(5) Calculate the width and depth of the strongest beam of rectangular section that can be cut out of a cylindrical log of wood whose diameter is 12″. (7″ wide × 10″ deep.)

(6) A cast iron test beam 1″ square in section and 2 feet long is freely supported at the ends. It fails with a central load of 500 lb-wt. What load at the free end will break a cantilever of the same material 2″ wide, 4″ deep and 5 feet long? (1600 lb-wt.)

(7) A test beam 1″ square in section is broken by a load of 150 lb-wt. applied at the centre of a span of 3 feet. Using a factor of safety of 8, calculate the safe uniformly distributed load for a beam 4½″ wide and 12″ deep freely supported over a span of 15 feet. (4860 lb-wt.)

(8) A beam is of solid circular section 6″ in diameter. Compare the weights of beams of equal strength of (i) hollow circular section with internal diameter $= \cdot 8 \times$ the external diameter and (ii) *I*-section 10″ deep and 5″ wide with uniform thickness of flange and web. {(i) ·5113 (ii) ·2415; thickness of flange and web $= \cdot 354''$ each.}

(9) The section of a beam is an isosceles triangle with base 8″ and side angles 55°30′. The neutral axis is parallel to the base. Calculate the moment of Inertia of the section about the N.A. and its section modulus. (43·80 in^4.; 11·29 in^3.)

(10) A 6″ × 4″ × ½″ unequal angle bar is used as a joist freely supported over a span of 10 feet with its longer leg placed vertically. Find the safe uniformly distributed load that it can carry if the maximum permissible bending stress is 7½ tons/in^2. Also calculate the maximum compressive stress induced under the load.

(·2162 ton per foot run; 3·72 tons/in^2.)

(11) A C.I. beam is of *I*-section with top flange 3″ × 1″ thick, bottom flange 8″ × 1½″ thick and the web 9½″ deep × ¾″ thick. Calculate the position of the neutral axis and the moment of Inertia of the section about the N.A. If the beam is freely supported over a span of 16 feet and carries a load of 5 tons uniformly spread over the span, calculate the skin stresses.

(3·98″ from base, 387·7 in^4.; $f_c = 2{\cdot}48$ tons/in^2.; $f_t = 1{\cdot}23$ tons/in^2.)

(12) A compound beam is made up of a 16″ × 6″ × 62 lb. R.S.J. with one 8″ × ½″ plate riveted to each flange. Find the safe distributed load that this beam will carry over a span of 30 feet if the maximum permissible stress is 7 tons/in^2. For the 16″ × 6″ × 62 lb. R.S.J. area of section $A = 18{\cdot}21$ in^2.; $I_{xx} = 725{\cdot}05$ in^4.; $I_{yy} = 27{\cdot}14$ in^4.

(23¼ tons.)

(13) A compound section is made up of one 24″ × 7½″ × 100 lb. R.S.J. with one 10″ × ½″ plate riveted to each flange. Calculate the modulus of section. For the 24″ × 7½″ × 100 lb. R.S.J., $I_{xx} = 2654$ in^4. (332·4 in^3.)

(14) A warehouse floor 72 feet × 30 feet is carried across main beams spaced 12 feet apart and subsidiary beams pitched $7\frac{1}{2}'$ centre to centre. The secondary beams are not continuous over the main beams but are connected to the web of the main beam on either side by cleats. If the floor load is 2 cwt. per square foot inclusive of the weights of the beams, select suitable sections for the beams if the permissible stress is 8 tons/in^2.

(Secondary, $8'' \times 5'' \times 28$ lb; main, $24'' \times 7\frac{1}{2}'' \times 100$ lb.)

(15) A stanchion is built-up of a $20'' \times 6\frac{1}{2}'' \times 65$ lb. R.S. Joist with a $10'' \times 6'' \times 40$ lb. R.S.J. riveted symmetrically on either side of the web to form the section of a cross [fig. 88 (*f*)]. Calculate the moments of Inertia of the section about the principal axes and the radii of gyration. For $20'' \times 6\frac{1}{2}'' \times 65$ lb. R.S.J., $A = 19{\cdot}12$ in^2.; $I_{xx} = 1226{\cdot}17$ in^4.; $I_{yy} = 32{\cdot}56$ in^4.; thickness of web $= {\cdot}45''$. For $10'' \times 6'' \times 40$ lb. R.S.J., $A = 11{\cdot}77$ in^2.; $I_{xx} = 204{\cdot}80$ in^4.; $I_{yy} = 21{\cdot}76$ in^4.

(1269·7 in^4.; 1084·8 in^4.; 5·46″; 5·04″.)

(16) A latticed stanchion is built-up of two $9'' \times 3''$ channels placed back to back at a distance of x inches apart [fig. 88 (*i*)]. What should be the value of x so that the section is equally strong in all directions? For each channel, $A = 5{\cdot}14$ in^2.; $P = {\cdot}78''$; $I_{xx} = 62{\cdot}52$ in^4.; $I_{yy} = 3{\cdot}75$ in^4. (5·2″.)

(17) A compound beam for a crane runway is built-up of a $22'' \times 7'' \times 75$ lb. R.S.J. with a $15'' \times 4''$ R.S. channel attached to its top flange [fig. 88 (*d*)]. Calculate the position of the N.A. of the section and the moment of Inertia about the N.A. For the $22'' \times 7'' \times 75$ lb. R.S.J., $A = 22{\cdot}06$ in^2.; $I_{xx} = 1676{\cdot}80$ in^4.; $I_{yy} = 41{\cdot}07$ in^4. For the $15'' \times 4''$ channel, $A = 10{\cdot}70$ in^2.; $P = {\cdot}97''$; $I_{xx} = 349{\cdot}10$ in^4.; $I_{yy} = 13{\cdot}34$ in^4.; thickness of web $= 0{\cdot}41''$.

(8″ from top; 2475·6 in^4.)

(18) Find the maximum stress in a compound girder built-up of three $15'' \times 5'' \times 42$ lb. R.S. Joists placed side by side with four $16'' \times \frac{1}{2}''$ flats riveted to the top and bottom flange (two on each) when carrying a load of 80 tons uniformly spread over a span of 25 feet. The weight of the girder is 250 lb. per foot run. I_{xx} for each joist is 428 in^4. (7·91 tons/in^2.)

(19) A compound girder consists of two $18'' \times 7'' \times 75$ lb. R.S. Joists placed side by side and four $18'' \times \frac{5}{8}''$ flats riveted to the flanges (two on top and two at bottom). For what maximum span may the girder be used to carry a load of 3 tons per foot run inclusive of its own weight if the maximum permissible bending stress is $7\frac{1}{2}$ tons/in^2.? I_{xx} for each joist $= 1149$ in^4. (32′6″.)

(20) A stanchion consists of two standard $12'' \times 3\frac{1}{2}''$ R.S. Channels placed $6\frac{1}{2}''$ apart, back to back and two $14'' \times \frac{1}{2}''$ flats riveted

at top and bottom to each flange [fig. 88(*c*)]. Calculate the I_{xx} and I_{yy} of the section and its least radius of gyration. For each 12″ × 3½″ R.S. Channel, $A = 7{\cdot}76$ in^2.; $P = {\cdot}83''$; $I_{xx} = 159{\cdot}73$ in^4.; $I_{yy} = 7{\cdot}15$ in^4. (1504·8 in^4.; 730 in^4.; 4·1″.)

(21) A 12″×6″×44 lb. R.S. Joist with thickness of flange ·72″ and thickness of web ·4″ has to resist a shear force of 20 tons at a certain cross-section. Calculate the maximum intensity of shear stress and its ratio to the average intensity. (4·77 tons/in^2.; 3·065.)

(22) A pipe-section with 6″ internal diameter and thickness of metal 1″, is subjected at a particular cross-section to a shear force of 30 tons. Find the maximum intensity of shear stress in the section. (2·7 tons/in^2.)

(23) A 6″ × 4″ × ½″ unequal angle section with the longer leg vertical is used as a beam and carries a load of ·2 ton per foot run over a span of 10 feet. Find the maximum intensity of shear stress and sketch the distribution of stress intensity across the section. (·4654 ton/in^2.)

(24) An 8″ × 3″ B.S. Channel with thickness of web ⅜″ and thickness of flanges ½″, is used as a beam with the 8″ base vertical. At a certain cross-section it has to resist a shear force of 10 tons. Calculate the maximum intensity of shear stress induced in the section and sketch the distribution of intensity across the section. (4 tons/in^2.)

(25) The section of a beam is a triangle with base *b* and height *h*, the base being placed horizontally. At a certain cross-section the shear force is *F*. Prove that the maximum intensity of shear stress occurs at $\frac{h}{2}$ and its magnitude is $\frac{3F}{bh}$ and that the shear intensity at the N.A. is $\frac{8F}{3bh}$.

(26) The triangular beam in Q. 9 carries at a certain section a shear force of 10 tons. Calculate the maximum and average intensities of shear stress. (·644 ton/in^2.; ·43 ton/in^2.)

(27) A 12″ × 5″ × 32 lb. R.S.J. has flanges ·55″ thick and web ·35″ thick. Sketch the distribution of shear intensity across the section and taking the maximum shear intensity as unity, calculate the values at 2″, 4″ and 5·45″ from the N.A. (·8511; ·7695; ·7518 in web; ·0526 in flange.)

(28) At a certain cross-section of a timber beam of rectangular section 6″ wide × 12″ deep, there is a bending moment of 144000 lb-inches and a shear force of 4320 lb-wt. Calculate the bending and shear intensities at 0, 2″, 4″ and 6″ from the N.A. Find the position of the principal planes and calculate the principal stresses at these points in the cross-section.

(29) A timber beam $4\frac{1}{2}''$ wide $\times$ $9''$ deep has two steel plates $8''$ deep $\times$ $\frac{5}{16}''$ thick firmly attached to it one on each side. The composite beam is loaded so that the skin-stress reached by timber is 1200 lb/in^2. What will be the maximum stress induced in the steel plates? Also calculate the total moment of resistance of the composite section at these stresses. Take $E_s = 20\ E_w$. (21333 lb/in^2.; 215122 lb-ins.)

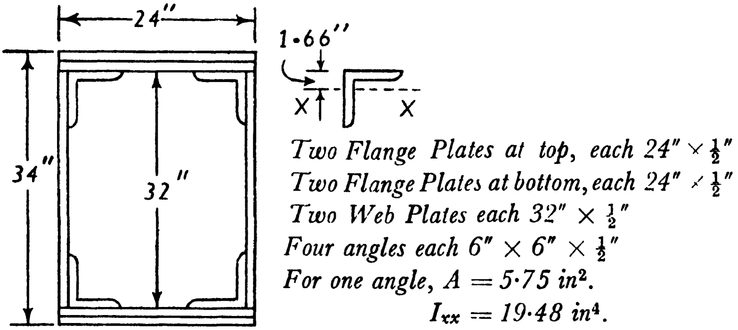
(30) A steel tube $1\frac{5}{8}''$ outside diameter and $1\frac{1}{4}''$ inside diameter is used as a simply supported beam on a span of 6 feet and it is found that the maximum safe load it can carry at mid-span is 280 lb.

Four of these tubes are placed parallel to one another and firmly fixed together to form in effect a single beam, the centres of the tubes forming a square of $1\frac{5}{8}''$ side with one pair of centres vertically over the other pair. Find the maximum central load which this beam can carry if the maximum stress is not to exceed that of the single tube above. (L.U.)

(1968 lb-wt.)

(31) A box section girder, 35 feet long, is simply supported at its ends and carries a uniformly distributed load over its whole length. If the section at mid-span is as shown in fig. 108(*a*), determine the greatest load which can be carried if the maximum allowable bending stress is $7\frac{1}{2}$ tons/in^2. Neglect the effect of rivet holes. (L.U.)

(4·95 tons per foot run.)



Two Flange Plates at top, each $24'' \times \frac{1}{2}''$
Two Flange Plates at bottom, each $24'' \times \frac{1}{2}''$
Two Web Plates each $32'' \times \frac{1}{2}''$
Four angles each $6'' \times 6'' \times \frac{1}{2}''$
For one angle, $A = 5{\cdot}75$ in^2.
$I_{xx} = 19{\cdot}48$ in^4.

FIG. 108(*a*)

(32) A composite beam consists of two timber beams of rectangular section each having a breadth B and a depth D together with a steel plate having a breadth (i.e. thickness) b and depth d. The plate is placed between the timber beams and all the three are joined together so that the whole section is symmetrical about the horizontal axis. If the maximum allowable stresses in the steel and timber are 18000 and 1000 lb/in^2. respectively and these are reached

simultaneously, find (a) the ratio of D and d and (b) the ratio of B and b in order that the moment of resistance of the timber alone shall be equal to that of the steel alone. E for steel $= 30 \times 10^6$ lb/in².; E for timber $= 1{\cdot}5 \times 10^6$ lb/in².

Make a dimensioned sketch of the section of the beam if the timber beams are each 10″ deep and 3″ wide and find the total moment of resistance of the beam. (L.U.)

[(a) 10:9. (b) 7·29: 1; 200000 lb-in.]

(33) A 14″ × 6″ B.S.B. has flanges 0·7″ thick and a web 0·4″ thick. At a certain section, the beam is subjected to a bending moment of 40 ton-feet and a shearing force of 18 tons. Due to the bending moment, the portion of the section above the N.A. is in compression.

Determine the principal stresses at a point 5″ above the neutral layer of the beam and the positions of the planes on which they act. Make a diagram to show clearly the stresses and planes and their positions. (L.U.)

$$\left\{\begin{array}{l}p_1 = 6{\cdot}928 \text{ tons/in}^2\text{.; compressive; } p_2 = 1{\cdot}452 \text{ ton/in}^2\text{., tensile;} \\ \text{Principal planes inclined at } 24°36', \ 114°36' \text{ to section.}\end{array}\right\}$$

(34) A beam of rectangular section 2″ wide and 6″ deep is subjected, at a certain section, to a bending moment which produces a maximum bending stress of 7½ tons/in². and to a shearing force which produces a maximum shearing stress of 3 tons/in². The bending moment produces compressive stresses above the neutral axis. Find the value of the tensile principal stress at the N.A. and at distances of 1″, 2″ and 3″ above and below the neutral axis.

Plot on squared paper a graph showing how the tensile principal stress varies across the section. (L.U.)

Above N.A.

y in inches	3	2	1	0
f in tons/in².	0	·505	1·695	3

Below N.A.

y in inches	0	1	2	3
f in tons/in².	3	4·195	5·505	7·5

(35) Two 9″ × 3″ channels are placed back to back and the top flanges only are connected by a plate ⅝″ thick and 9″ wide. This compound section forms a simply supported beam 5 feet long and carries a uniformly distributed load of 8 tons per foot-run. If the flange is connected to the channels by ¾″ diameter rivets, calculate the pitch in order that the shearing stress in them should not exceed 5 tons/in². I_{max} for one channel $= 134{\cdot}8$ in⁴; $A = 5{\cdot}86$ in². (L.U.)

($p = 4{\cdot}303''$, say 4″.)

(36) A rectangular wooden beam 2″ wide and 6″ deep is reinforced by screwing a steel plate ¼″ thick and 2″ wide on to the bottom. The screws are ¼″ diameter and pitched 3″ apart. They are a close fit in the plate.

The beam is simply supported at the ends over a span of 8 feet and is loaded at the centre by a load of 2 cwt.

Calculate the maximum stresses in the steel plate and timber and the maximum shearing stress in the screws.

Neglect the weight of the beam itself and any weakening of the plate due to the screw holes. Take the modular ratio of steel and timber as 15. (L.U.)

$$\left\{\begin{array}{l}2040\ \text{lb/in}^2.\ \text{in plate};\ 278{\cdot}6\ \text{lb/in}^2.\ \text{in timber};\\ 1220\ \text{lb/in}^2.\ \text{shear stress in screws.}\end{array}\right\}$$

(37) A bar of regular hexagonal section of side a is subjected to a shearing force S which acts at right angles to one of the diagonals of the hexagon. Derive a formula for the shearing stress q at a distance x from the diagonal and hence find the ratio of q to the mean shearing stress for the particular values $x=0$ and $x=\frac{a}{2}$.

Indicate by means of a sketch how the shearing stress varies across the depth of the section. (L.U.)

$$\left\{\begin{array}{l}\text{for } x=0,\ 1{\cdot}2\\ \text{for } x=\frac{a}{2},\ 1{\cdot}006.\end{array}\right\}$$

CHAPTER VI

BEAMS AND BENDING — III.

Deflections of Beams

1. Strength and stiffness of beams: In the previous chapter, we have studied the resistance set up by a beam-section to the effects of the load system on it. For this purpose, we made use, mainly, of the relation $\frac{p}{y} = \frac{M}{I}$ in the key formula $\frac{M}{I} = \frac{p}{y} = \frac{E}{R}$. We shall now turn our attention from the "strength" of a beam to its "stiffness" and to that end, we shall look a little more closely at the relation $\frac{M}{I} = \frac{E}{R}$. Under transverse loading a beam bends and it is necessary to study the deflection undergone by the centre-line of the beam at any point in its length and particularly the maximum deflection because just as the maximum skin-stress induced must not exceed the working stress of the material, the maximum deflection will also have to be limited i.e. it must not exceed a certain permissible limit in relation to the span of the beam. The beam has to be designed not merely from the point of view of "strength" but also "stiffness" which is inversely proportional to its deflection. We shall therefore devote some time to study the deflection of a beam at any point in its centre line, which in effect comes to investigating the equation of the curve which the beam takes up under a given system of loading on a given span. In the following articles, we shall investigate the slope and deflection at any point in the centre line of a beam.

2. Bending into a circular arc: If a beam AB of length l bends into an arc ADB of radius R, the central deflection δ is easily calculated. Referring to fig. 109, $AC.CB =$

CD.CE, from the well-known property of intersecting chords of a circle.

$$\therefore \quad \frac{l}{2}\cdot\frac{l}{2} = \delta(2R - \delta)$$

$$\text{or} \quad -\delta^2 + 2R.\delta = \frac{l^2}{4}.$$

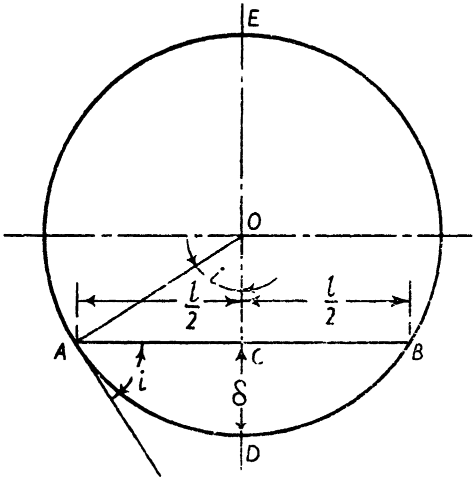


FIG. 109

Since δ is usually very small δ^2 is a small, quantity of the second order and may be neglected.

$$\therefore \quad 2R.\delta = \frac{l^2}{4}$$

$$\text{and} \quad \delta = \frac{l^2}{8R}.$$

The maximum slope i at A or B is equal to angle AOC or BOC in the figure.

$\therefore \quad \sin i = \frac{AC}{AO} = \frac{l}{2R}$. Since i is also a small quantity,

$\sin i = i$ in radians for all practical purposes.

$$\therefore \quad i = \frac{l}{2R} \text{ radian.}$$

Beams bend to an arc of a circle only in a few cases. In the simple case of a simply supported beam carrying a central point-load, it will be seen later that the equation of the curve it takes up is a cubic and for uniformly distributed loads on it, it is a quartic. One obvious case in which the beam bends to an arc of a circle is that of a beam of uniform section subjected to a constant bending moment M at all sections. In the relation $\frac{M}{I} = \frac{E}{R}$, M, I and E being all constant in this case, R is constant, i.e., the beam bends into a circular arc of radius $R = \frac{EI}{M}$. The maximum deflection at the centre will be $\delta = \frac{l^2}{8R} = \frac{M.l^2}{8EI}$, the maximum slope at the supports will be $i = \frac{l}{2R} = \frac{M.l}{2EI}$ radians.

Another important case in which the beam bends to a circular arc may be noted. The radius of curvature at any point in the centre line of the beam is usually variable because the bending moment M varies from section to section. However, if the moment of Inertia I of the section is so selected to suit the bending moment M on it that $\frac{M}{I}$ is a constant, we shall have a case in which both M and I vary and yet their ratio is constant. In this case, $\frac{M}{I} = \frac{E}{R} =$ a constant. Therefore R is a constant i.e. the beam bends to an arc of a circle. This condition is realised in a beam of "Uniform strength', of constant depth d. For in this case, $\frac{M}{I} = \frac{p}{y} = \frac{f}{d/2} = \frac{E}{R}$ $\therefore$ $R = \frac{E.d}{2f}$, a constant.

The central deflection $\delta = \frac{l^2}{8R} = \frac{fl^2}{4E.d}$ and the maximum slope $i = \frac{l}{2R} = \frac{f.l}{E.d}$ radian for a beam of constant depth d having a uniform skin stress f at all sections.

3. Relation between slope, deflection and radius of curvature: Consider an element $PQ = \delta s$ of a curve referred to OX and OY as axes of reference. Let Ψ and $\Psi + \delta\Psi$ be the slopes at P and Q respectively. Then $\angle PO'Q$ in fig. 110 is $\delta\Psi$ and $O'P = O'Q = R$ the radius of curvature at P, $O'P$ and $O'Q$ being normals to the curve at P and Q.

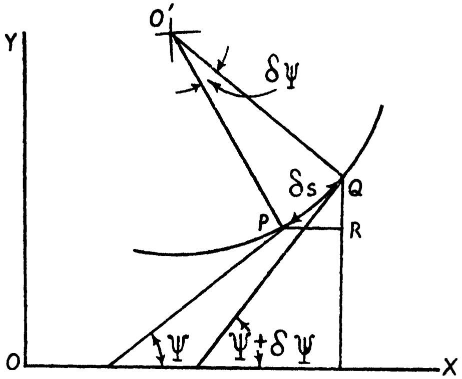


FIG. 110

Now, $PQ = \delta s = R.\ \delta\Psi$, i.e. $ds = R.d\Psi$ .... (1)

In the figure, let $PR = \delta x$ and $RQ = \delta y$, then since $PQ^2 = PR^2 + RQ^2$, we get,

$$\delta s^2 = \delta x^2 + \delta y^2$$

$$\therefore \quad \delta s = \sqrt{\delta x^2 + \delta y^2}, \text{ or } ds = \sqrt{dx^2 + dy^2}$$

Also $\tan\Psi = \dfrac{RQ}{PR} = \dfrac{dy}{dx}$

Differentiating,

$$\sec^2\Psi.d\Psi = \frac{d^2y}{dx^2}.dx$$

$$\therefore \quad (1 + \tan^2\Psi)\ d\Psi = \frac{d^2y}{dx^2}.dx$$

or $$\left\{1 + \left(\frac{dy}{dx}\right)^2\right\} d\Psi = \frac{d^2y}{dx^2}.\ dx \text{ i.e. } d\Psi = \frac{\frac{d^2y}{dx^2}.dx}{I + \left(\frac{dy}{dx}\right)^2}$$

Substituting for ds and $d\Psi$ in (1) we get,

$$\sqrt{dx^2 + dy^2} = \frac{R \times \frac{d^2y}{dx^2}.dx}{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}}$$

$$\therefore \quad dx\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \frac{R.\dfrac{d^2y}{dx^2}dx}{\left\{1 + \left(\dfrac{dy}{dx}\right)^2\right\}}$$

The radius of curvature is therefore given in cartesian coordinates by the relation,

$$R = \frac{\left\{1 + \left(\dfrac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}}{\dfrac{d^2y}{dx^2}}.$$

Substituting for R in the equation $\frac{M}{I} = \frac{E}{R}$ or $M = \frac{EI}{R}$ we get,

$$M = \frac{EI\dfrac{d^2y}{dx^2}}{\left\{1 + \left(\dfrac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}}$$

or

$$M\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}} = EI\frac{d^2y}{dx^2}.$$

Since $\frac{dy}{dx}$ (the tangent of the slope at any point in the centre line of the bent beam) is an extremely small quantity, $\left(\frac{dy}{dx}\right)^2$ is much smaller still and may be neglected. The expression $\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}$ is therefore equal to unity, to all intents and purposes, and we get the simple relation,

$$\boxed{M = EI\frac{d^2y}{dx^2}} \quad \text{.......................................} \quad (2)$$

The relation between the moment of resistance and therefore the bending moment at a section and the cartesian coordinates of the point in the bent beam is the differential equation,

$$EI\frac{d^2y}{dx^2} = M.$$

Since we know the bending moment M at any section of a beam with a given loading, one integration of this expression will enable us to evaluate $\frac{dy}{dx}$ or slope of the beam at the section and a second integration will help us to get the deflection y if M is an integrable function of x.

We shall now apply the basic formula $EI\frac{d^2y}{dx^2} = M$ to a few standard cases, but before doing so, we must select suitable axes of reference.

4. Axes of reference: In studying the deflections of a beam if we take the origin at the left hand support and x positive to the right, the deflected beam will usually lie in the fourth quadrant and the deflection y will be negative according to the standard cartesian convention. Most of the cases

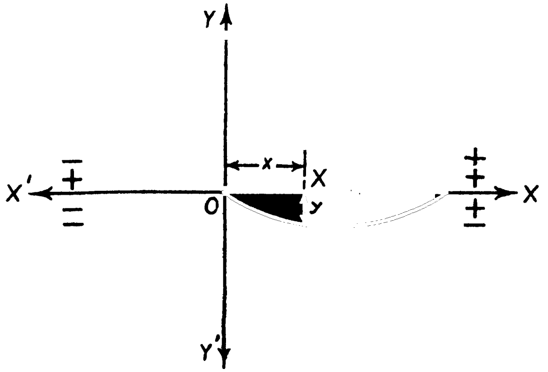


FIG. 111

of beam-deflections which we shall be studying will come under this class. It is therefore convenient to alter the convention to suit our requirements.

For the beam shown in fig. 112, take the origin at A. We shall consider x positive to the *right* and y positive *downwards*.

Let (x, y) be the coordinates of a point P on the centre line of the deflected beam and $(x + \delta x,\ y + \delta y)$ those of a contiguous point Q. For the beam which is bent as shown in the figure, δy and δx are both positive. $\dfrac{dy}{dx}$ and therefore the slope i at P is thus positive. If $(i + \delta i)$ be the slope at Q, it is less than the slope i at P. Therefore δi is negative and $\dfrac{di}{dx}$ is also negative. If M is the bending moment at the section at P, $M = EI\dfrac{d^2y}{dx^2} = EI.\dfrac{di}{dx}$. Since $\dfrac{di}{dx}$ is negative, M is negative.

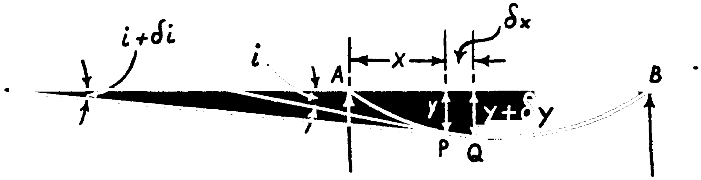


FIG. 112

We shall therefore adopt the convention that a bending moment which produces concavity (of the centre line of a beam) upwards, will be marked negative. Then a bending moment which produces convexity upwards will, ipso facto, be marked positive. With this convention when the expression $EI\dfrac{d^2y}{dx^2} = M$ is integrated twice, the value of y so obtained will be positive.

5. Cantilevers: (i) Span l; a point load W at the free end.

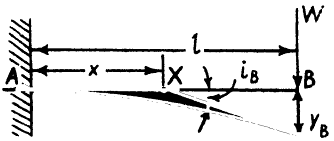


FIG. 113

Take the origin at the fixed end A and consider a section X distant x from the origin.

$$M_x = + W(l - x)$$

$$\therefore\quad EI\frac{d^2y}{dx^2} = + W(l - x)$$

Integrating,

$$EI\int\frac{d^2y}{dx^2}.\ dx=\int W\,(l-x)\ dx.$$

$\therefore$ $$EI\frac{dy}{dx}=W\,(lx-\frac{x^2}{2})+C_1$$ where C_1 is a constant of integration.

This expression will give us the slope at any section X as soon as we evaluate the constant C_1. At the fixed end A the slope must be zero.

Therefore at $x=0$, $\frac{dy}{dx}=0$. Substituting, we get, $C_1=0$.

The expression for slope is therefore,

$$EI\frac{dy}{dx}=W\,(lx-\frac{x^2}{2}).$$

Integrating once more. we get,

$$EI\int\frac{dy}{dx}\,dx=\int W\,(lx-\frac{x^2}{2})\ dx$$

$$\therefore\qquad EIy=W\ \ (l\,\frac{x^2}{2}-\frac{x^3}{6}\,)+C_2,$$

where C_2 is a constant of integration.

At A, $x=0$ and $y=0$.

Substituting these values in the expression, we get $C_2=0$.

The expression for deflection at any point is therefore,

$$EIy=W\,(l\,\frac{x^2}{2}-\frac{x^3}{6}).$$

We are thus able to get the slope and deflection at any point in the centre line of the cantilever. The maximum slope and deflection occur at the free end. Using the letter i for the angle of inclination or slope and remembering that $\tan i=i$ for very small angles, we get,

$$EIi_B=W\,(l^2-\frac{l^2}{2})=\frac{Wl^2}{2},\ \text{putting } x=l.$$

$$\therefore\qquad i_B=\frac{Wl^2}{2EI}\ \text{radians.}$$

Similarly, $EIy_B = W\left(\frac{l^3}{2} - \frac{l^3}{6}\right) = \frac{Wl^3}{3}$

$\therefore \qquad y_B = \frac{Wl^3}{3EI}.$

(ii) Cantilever with uniformly distributed load w over span l.

Take the origin at the fixed end A. At a section X, distant x from the origin,

$$EI\frac{d^2y}{dx^2} = +\frac{w(l-x)^2}{2}.$$

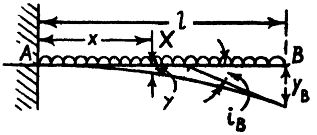


FIG. 114

Integrating,

$$EI\frac{dy}{dx} = -\frac{w}{6}(l-x)^3 + C_1$$

At $\quad x = 0, \frac{dy}{dx} = 0 \quad \therefore 0 = -\frac{wl^3}{6} + C_1$

or $\quad C_1 = \frac{wl^3}{6}$

$$\therefore \quad EI\frac{dy}{dx} = -\frac{w}{6}(l-x)^3 + \frac{wl^3}{6}$$

Integrating once more,

$$EIy = +\frac{w}{24}(l-x)^4 + \frac{wl^3}{6}.x + C_2$$

At $\quad x = 0, y = 0 \quad \therefore 0 = \frac{wl^4}{24} + C_2$

or $\quad C_2 = -\frac{wl^4}{24}.$

The expression for deflection is therefore,

$$EIy = +\frac{w}{24}(l-x)^4 + \frac{wl^3}{6}.x - \frac{wl^4}{24}.$$

The maximum values of slope and deflection are at the free end. Substituting $x = l$, we get,

$$EIi_B = \frac{wl^3}{6} \text{ or } i_B = \frac{wl^3}{6EI}$$

$$\text{and } EIy_B = \frac{wl^4}{6} - \frac{wl^4}{24} = \frac{wl^4}{8} \text{ or } y_B = \frac{wl^4}{8EI}.$$

If W is the total load, $W = wl$ and the expressions in terms of the total load become,

$$i_B = \frac{Wl^2}{6EI} \text{ and } y_B = \frac{Wl^3}{8EI}.$$

(iii) Cantilever with a point-load W, not at the free end.

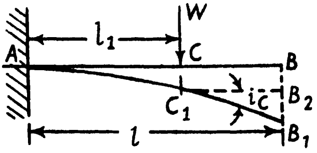


FIG. 115

Let W be applied at C at a distance l_1 from the fixed end A of a cantilever of span l. Let i_C be the slope and CC_1 the deflection at C. The maximum deflection will occur at the free end B. Since there is no bending moment anywhere in CB, there will be no curvature, the slope anywhere in CB remaining constant and equal to i_C. CB will take up the position C_1B_1 in the deflected cantilever. Draw C_1B_2 perpendicular to BB_1. Then the deflection at B is,

$$y_B = BB_1 = BB_2 + B_2B_1 = CC_1 + B_2B_1 = y_C + (l - l_1)i_C$$

$$= \frac{Wl_1^3}{3EI} + (l - l_1)\frac{Wl_1^2}{2EI}$$

$$\text{and } i_B = i_C = \frac{Wl_1^2}{2EI}.$$

(iv) Cantilever partially loaded with w from fixed end. The slope and deflection at C in fig. 116 are,

$$i_C = \frac{wl_1^3}{6EI} \text{ and } y_C = \frac{wl_1^4}{8EI}$$

$$
\begin{aligned}
y_B &= y_C + (l - l_1)\, i_C \\
&= \frac{wl_1^4}{8EI} + (l - l_1).\frac{wl_1^3}{6EI}.
\end{aligned}
$$

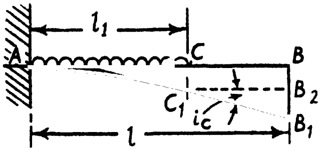


FIG. 116

The slope throughout from C to B is constant and equal to i_C i.e. $\frac{wl_1^3}{6EI}$.

(v) Cantilever partially loaded with w from free end.

The deflection at any point, in this case, may be looked upon as the difference of deflections, at the point, with

(a) the entire span loaded from A to B and

(b) the portion from A to C only loaded.

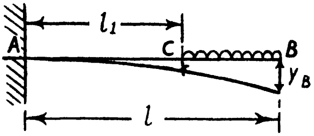


FIG. 117

The maximum deflection will occur at the free end, being given by,

$$y_B = \frac{wl^4}{8EI} - \left\{\frac{wl_1^4}{8EI} + (l - l_1)\frac{wl_1^3}{6EI}\right\}$$

Example 1.

A cantilever 5 feet long is loaded with a point load of 150 lb. at the free end and a distributed load of 120 lb. per foot run over 3 feet from the fixed end as shown in fig. 118. If the section is rectangular 3″ × 6″ deep, calculate the deflection at the free end. $E = 1{\cdot}5 \times 10^6$ *lb/in*^{2}.

For the section,
$$
\begin{aligned}
I &= \tfrac{1}{12}\, bd^3 \\
&= \tfrac{1}{12} \times 3 \times 6^3 \\
&= 54 \text{ in}^4.
\end{aligned}
$$

The deflection at B due to the point load is,

$$y'_B = \frac{Wl^3}{3EI} = \frac{150 \times 5^3}{3EI} = \frac{6250}{EI}$$

The deflection at B, due to the distributed load on AC is,

$$y''_B = \frac{Wl_1^3}{8EI} + (l - l_1)\frac{Wl_1^2}{6EI} = \frac{360}{EI}\left\{\frac{27}{8} + \frac{2 \times 3^2}{6}\right\} = \frac{2295}{EI}.$$

The total deflection at B is,

$$y_B = y_B' + y_B'' = \frac{6250}{EI} + \frac{2295}{EI} = \frac{8545}{EI}.$$

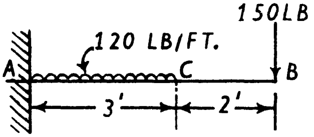


FIG. 118

Important note:

In getting this result, we have used feet units for lengths. We shall have to use feet units for both E and I and the deflection will work out in feet units. To bring it to inch-units, we shall have to multiply the result by 12. Usually, spans of beams are given in feet and E and I are in inch-units. Therefore, substituting for E and I in *inch-units* in the expression for y_B and multiplying the result by 12^3 or 1728, will give us the deflection y_B in inch-units. This makes all numerical work in deflections easy and confusion of units is avoided.

Proceeding with the example and substituting for E and I in inch-units, we get,

$$y_B = \frac{8545 \times 1728}{1{\cdot}5 \times 10^6 \times 54} = {\cdot}1823''.$$

6. Propped Cantilevers: A cantilever carrying a uniformly distributed load w per foot run over its entire span is propped at the free end to the level of the fixed end. To find the reaction P of the prop, imagine the prop removed. The free end B will deflect $\frac{wl^4}{8EI}$ under the load. The effect of introducing the prop is to exert an upward force P at the free end and the upward deflection of B due to P will neutralise the downward deflection due to the load, if the prop is at the same level as the fixed end A. The upward deflection due to the point-load P at B will be $\frac{Pl^3}{3EI}$.

Equating, $\frac{Pl^3}{3EI} = \frac{wl^4}{8EI}$

$\therefore \quad P = \frac{3}{8}wl.$

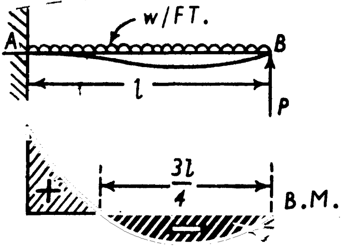


FIG. 119

The B.M. diagram for the propped cantilever is shown in fig. 119. Taking the origin at B, consider a section X at a distance x from B.

Then $M_x = -P.x + \frac{wx^2}{2}$

$= -\frac{3}{8}wlx + \frac{wx^2}{2}$

This is zero at $x = 0$ and at $x = \frac{3l}{4}$, the latter being a point of contraflexure. The maximum negative bending moment is at $x = \frac{3l}{8}$ and is given by,

$M_{max} = -\frac{3}{8}wl.\,\frac{3}{8}l + \frac{w}{2}.\,\frac{9l^2}{64}$

$= -\frac{9}{128}wl^2.$

The maximum positive value occurs at the fixed end and is,

$$M_{max} = -\tfrac{3}{8}wl.l + \frac{wl^2}{2} \text{ or } + \frac{wl^2}{8}.$$

For slope and deflection at a section,

$$M_x = EI\frac{d^2y}{dx^2} = -\tfrac{3}{8}\,wlx + \frac{wx^2}{2}$$

Integrating,

$$EI\frac{dy}{dx} = -\frac{3}{16}\,wlx^2 + \frac{wx^3}{6} + C_1.$$

At A, $x = l$ and $\frac{dy}{dx} = 0$. Substituting, we get,

$$0 = -\frac{3wl^3}{16} + \frac{wl^3}{6} + C_1$$

$$\therefore \quad C_1 = \frac{wl^3}{48}$$

$$\therefore \quad EI\frac{dy}{dx} = -\frac{3}{16}wlx^2 + \frac{wx^3}{6} + \frac{wl^3}{48} \qquad \ldots\ldots\ldots\ldots(1)$$

Integrating once more,

$$EIy = -\frac{1}{16}wlx^3 + \frac{wx^4}{24} + \frac{wl^3}{48}.x + C_2.$$

At B, $x = 0$ and $y = 0$ $\therefore C_2 = 0$

$$\therefore \quad EIy = -\frac{1}{16}wlx^3 + \frac{wx^4}{24} + \frac{wl^3}{48}.x \qquad \ldots\ldots\ldots\ldots(2)$$

Relations (1) and (2) give the values of slope and deflection at any section of the beam from B to A.

If the prop at the free end is δ units lower than the level of the fixed end, P has to neutralise a downward deflection of $\left(\frac{wl^4}{8EI} - \delta\right)$ only. Equating $\frac{Pl^3}{3EI}$ to $\left(\frac{wl^4}{8EI} - \delta\right)$, the magnitude of P for a given difference of levels δ between the supports can be obtained.

Example 2.

A cantilever of span l is supported at the free end to the level of the fixed end. It carries a concentrated load W at the centre of the span. Calculate the reaction of the prop and construct the S.F. and B.M. diagrams for the beam. Also find the position and amount of maximum deflection.

If the prop is removed the downward deflection of B would be,

$$\frac{W}{3EI}\left(\frac{l}{2}\right)^3 + \frac{l}{2}\cdot\frac{W}{2EI}\left(\frac{l}{2}\right)^2 \text{ i.e. } \frac{Wl^3}{24EI} + \frac{Wl^3}{16EI} \text{ or } \frac{5Wl^3}{48EI}.$$

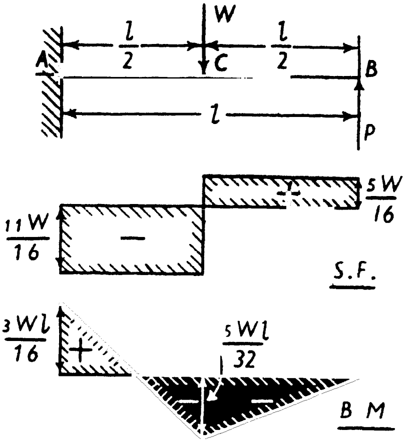


Fig. 120

The upward deflection due to P has to neutralise this

$$\therefore \quad \frac{Pl^3}{3EI} = \frac{5Wl^3}{48EI}$$

or $\quad P = \frac{5}{16}W.$

The bending moment at C is,

$$M_C = -P \times \frac{l}{2} = -\frac{5Wl}{32}$$

The bending moment at A is,

$$M_A = -P.l + W.\frac{l}{2} = -\frac{5Wl}{16} + \frac{Wl}{2} = +\frac{3Wl}{16}.$$

The S.F. and B.M. diagrams are shown in fig. 120. For the position and amount of maximum deflection, take the origin at B and consider a section X in CB at a distance x from B i.e. $x < \frac{l}{2}$.

Then $EI\frac{d^2y}{dx^2} = -P.x = -\frac{5Wx}{16}$

Integrating,

$$EI\frac{dy}{dx} = -\frac{5W.x^2}{32} + C_1.$$

At $x = \frac{l}{2}, \frac{dy}{dx} = i_C$

$$\therefore \quad EIi_C = -\frac{5Wl^2}{128} + C_1$$

or $C_1 = \frac{5Wl^2}{128} + EIi_C$

$$\therefore \quad EI\frac{dy}{dx} = -\frac{5Wx^2}{32} + \frac{5Wl^2}{128} + EIi_C \ldots\ldots (1)$$

Integrating once more

$$EIy = -\frac{5Wx^3}{96} + \frac{5Wl^2x}{128} + EIi_C.x + C_2$$

At B, $x = 0$ and $y = 0$ $\therefore C_2 = 0$

$$\therefore \quad EIy = -\frac{5Wx^3}{96} + \frac{5Wl^2x}{128} + EIi_C.x \ldots\ldots (2)$$

Relations (1) and (2) will give us the slope and deflection at any section between B and C, if i_C is evaluated.

To evaluate i_C consider a section X in CA, at a distance of x from B, i.e. $x > \frac{l}{2}$.

Now, $EI\frac{d^2y}{dx^2} = -P.x + W\left(x - \frac{l}{2}\right) = -\frac{5}{16}Wx + W\left(x - \frac{l}{2}\right)$

Integrating,

$$EI\frac{dy}{dx} = -\frac{5}{32}Wx^2 + \frac{W}{2}\left(x - \frac{l}{2}\right)^2 + C'_1.$$

At A, $x = l$ and $\frac{dy}{dx} = 0.$

Substituting, $0 = -\frac{5Wl^2}{32} + \frac{Wl^2}{8} + C'_1$

$$\therefore \quad C'_1 = \frac{Wl^2}{32}$$

$$\therefore \quad EI\frac{dy}{dx} = -\frac{5}{32}Wx^2 + \frac{W}{2}\left(x-\frac{l}{2}\right)^2 + \frac{Wl^2}{32}$$

This expression gives the slope at any section between C and A. At C, $x = \frac{l}{2}$ and $\frac{dy}{dx} = i_C$.

$$\therefore \quad EIi_C = -\frac{5}{32}\cdot\frac{Wl^2}{4} + \frac{Wl^2}{32} = -\frac{Wl^2}{128}$$

Substituting for EIi_C in (1) and (2) above, the slope and deflection for any section between B and C are given by,

$$EI\frac{dy}{dx} = -\frac{5Wx^2}{32} + \frac{5Wl^2}{128} - \frac{Wl^2}{128} = -\frac{5Wx^2}{32} + \frac{Wl^2}{32}$$

$$\text{and} \quad EIy = -\frac{5Wx^3}{96} + \frac{5Wl^2.x}{128} - \frac{Wl^2}{128}.x$$

$$= -\frac{5Wx^3}{96} + \frac{Wl^2x}{32} = \frac{Wx}{96}\,(3l^2 - 5x^2)$$

The maximum deflection occurs at the section where

$$\frac{dy}{dx} = 0$$

$$\therefore \quad -\frac{5Wx^2}{32} + \frac{Wl^2}{32} = 0$$

or $\quad 5x^2 = l^2$

and $\quad x = \frac{l}{\sqrt{5}}$ from B.

$$EIy_{max} = \frac{W}{96}\cdot\frac{l}{\sqrt{5}}\,(3l^2 - l^2) = \frac{Wl^3}{48\sqrt{5}}$$

$$\therefore \quad y_{max} = \frac{Wl^3}{48\sqrt{5}EI}.$$

In problems with point-loads, a point like C which is common to both BC and CA, is very useful for evaluating the constants of integration. We made use of the law of slopes for the portion CA to evaluate i_C and then substituted this value in the expressions (1) and (2) for slope and deflection in the portion BC.

There is an elegant method to solve such problems with ease. It is described later in art. 11 of this chapter.

7. Simply supported beams: (i) Span *l*; Central load *W*.

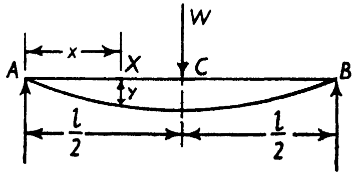


FIG. 121

Take the origin at *A* and consider a section *X* at a distance of *x* from *A*, in the portion *AC*.

Then,

$$EI\frac{d^2y}{dx^2} = -\frac{W}{2}.x.$$

Integrating,

$$EI\frac{dy}{dx} = -\frac{Wx^2}{4} + C_1.$$

At *C*, by symmetry, the slope is zero.

$$\therefore \quad \text{At } x = \frac{l}{2}, \ \frac{dy}{dx} = 0; \text{ substituting,}$$

$$0 = -\frac{Wl^2}{16} + C_1$$

$$\text{or} \quad C_1 = \frac{Wl^2}{16}$$

$$\therefore \quad EI\frac{dy}{dx} = -\frac{Wx^2}{4} + \frac{Wl^2}{16}.$$

Integrating once more,

$$EIy = -\frac{Wx^3}{12} + \frac{Wl^2}{16}.x + C_2$$

At A, $x = 0$ and $y = 0$ $\therefore C_2 = 0$.

The law of deflection between *A* and *C* is,

$$EIy = -\frac{Wx^3}{12} + \frac{Wl^2}{16}.x.$$

The maximum slope is at A and is given by,

$$EIi^{A} = +\frac{Wl^2}{16}$$

$$\therefore \quad i_A = \frac{Wl^2}{16EI}. \quad \text{By symmetry, } i_B = \frac{Wl^2}{16EI}.$$

The maximum deflection is at the centre.

$$EIy_C = -\frac{Wl^3}{96} + \frac{Wl^3}{32} = \frac{Wl^3}{48}$$

$$\therefore \quad y_C = \frac{Wl^3}{48EI}.$$

(ii) Span l; uniformly distributed load w per foot run over the entire span.

Take the origin at A and consider a section X at a distance x from the origin.

$$EI\frac{d^2y}{dx^2} = -\frac{wl}{2}.x + \frac{wx^2}{2}$$

Integrating,

$$EI\frac{dy}{dx} = -\frac{wlx^2}{4} + \frac{wx^3}{6} + C_1$$

At $\quad x = \frac{l}{2}, \; \frac{dy}{dx} = 0$

$$\therefore \quad 0 = -\frac{wl^3}{16} + \frac{wl^3}{48} + C_1$$

$$\therefore \quad C_1 = \frac{wl^3}{24}$$

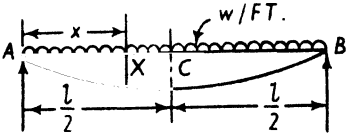


FIG. 122

The law of slopes is, therefore,

$$EI\frac{dy}{dx} = -\frac{wlx^2}{4} + \frac{wx^3}{6} + \frac{wl^3}{24}.$$

Integrating once more,

$$EIy = -\frac{wlx^3}{12} + \frac{wx^4}{24} + \frac{wl^3}{24}.x + C_2$$

At A, $x = 0$ and $y = 0$ $\therefore$ $C_2 = 0$.

The law of deflections is,

$$EIy = \quad \frac{wlx^3}{12} + \frac{wx^4}{24} + \frac{wl^3}{24}.x.$$

The maximum slope is at the supports,

$$EIi_A = \frac{wl^3}{24}$$

$$\therefore \quad i_A = \frac{wl^3}{24EI} = \frac{Wl^2}{24EI},$$ where W is the total load.

By symmetry, $i_B = -\dfrac{wl^3}{24EI} = -\dfrac{Wl^2}{24EI}$.

At the centre $x = \dfrac{l}{2}$ and the maximum deflection is given by,

$$EIy_C = -\frac{wl^4}{96} + \frac{wl^4}{384} + \frac{wl^4}{48} = \frac{(-4 + 1 + 8)\ wl^4}{384}$$

$$= \frac{5}{384}\ wl^4$$

$$\therefore \quad y_C = \frac{5}{384}.\frac{wl^4}{EI} = \frac{5}{384}.\frac{W.l^3}{EI}$$ in terms of the total load W.

Example 3.

A timber beam of rectangular section 4″ wide and 9″ deep is simply supported over a span of 12 feet. What uniformly distributed load should the beam carry to produce a central deflection of $\frac{1}{4}$″*? Take E = 1·5 × 10*^{6} *lb/in*^{2}*.*

For the section, $I = \frac{1}{12}\,bd^3 = \frac{1}{12} \times 4 \times 9^3 = 243$ in^4.

$$y_C = \frac{5}{384}.\frac{W.l^3}{EI} = \frac{5}{384} \times \frac{W.12^3 \times 1728}{1{\cdot}5 \times 10^6 \times 243} = \frac{1}{4}''$$

$\therefore$ $W = 2344$ lb-wt.

Example 4.

A girder of uniform section and constant depth of 6 feet is freely supported over a span of 60 feet. Calculate the deflection for a uniformly distributed load on it such that the maximum bending stress induced is $7\frac{1}{2}$ tons/in^2. $E = 13000$ tons/in^2.

If, for the above girder, the flange areas are so proportioned that there is a uniform flange stress of $7\frac{1}{2}$ tons/in^2. throughout the beam, calculate the central deflection.

The central deflection for a girder of uniform section is

$$y_c = \frac{5}{384}\cdot\frac{Wl^3}{EI}$$

The load W is such that the maximum skin stress is $7\frac{1}{2}$ tons/in^2. Now, the maximum bending moment at the centre is $= \frac{W.l}{8} = \frac{W \times 60 \times 12}{8}$

$= 90W$ ton-inches.

The moment of resistance $= f.Z = f \times \frac{I}{d/2}$

$$= \frac{15}{2} \times \frac{I}{36} = \frac{5I}{24} \text{ ton-inches.}$$

Equating,

$$90W = \frac{5I}{24}$$

or $\quad W = \frac{I}{432}$ tons.

Substituting for W in the expression for y_c.

$$y_c = \frac{5}{384} \times \frac{I}{432} \times \frac{60^3}{13000 \times I} \times 1728 = \frac{45}{52} = \cdot 865''.$$

If the girder is of constant depth and uniform strength, it bends to an arc of a circle of radius R given by,

$$\frac{f}{d/2} = \frac{E}{R} \therefore R = \frac{E.d}{2f} = \frac{13000 \times 72}{2 \times 7\cdot 5} = \frac{13000 \times 72}{15} \text{ inches.}$$

The central deflection will now be given by,

$$\delta = \frac{l^2}{8R} = \frac{720 \times 720 \times 15}{8 \times 13000 \times 72} = \frac{27}{26} = 1\cdot 04''.$$

8. Relation between maximum stress and maximum deflection: (i) Simply supported beam of span l, central load W.

Let f and δ be the maximum stress and maximum deflection due to the load.

$$\text{Then } \frac{Wl}{4} = f \cdot Z = f \times \frac{I}{d/2} = \frac{2fI}{d} \qquad \text{(1)}$$

$$\text{Also} \quad \delta = \frac{Wl^3}{48EI} \qquad \text{(2)}$$

Eliminating W between (1) and (2), we get,

$$\delta = \frac{l^2}{48EI} \times \frac{8f.I}{d} = \frac{fl^2}{6E.d}$$

This can be rearranged as,

$$\frac{\delta}{l} \times \frac{d}{l} = \frac{1}{6} \times \frac{f}{E} \qquad \text{(A)}$$

This is a convenient form, in which, $\frac{d}{l}$ is the ratio of depth to span and $\frac{\delta}{l}$ is the ratio of maximum deflection to span.

(ii) Simply supported beam of span l with a uniformly distributed load of w per foot run.

$$\text{In this case, } \frac{Wl}{8} = fZ = \frac{2fI}{d} \qquad \text{(3)}$$

$$\text{And} \quad \delta = \frac{5}{384} \times \frac{Wl^3}{EI} \qquad \text{(4)}$$

Eliminating W between (3) and (4) we get,

$$\delta = \frac{5}{384} \times \frac{l^2}{EI} \times \frac{16f.I}{d} = \frac{5}{24} \cdot \frac{f.l^2}{E.d}$$

This can be rearranged as,

$$\frac{\delta}{l} \times \frac{d}{l} = \frac{5}{24} \times \frac{f}{E} \qquad \text{(B)}$$

(iii) Beam of uniform section, span l, subjected to a constant bending moment M.

This beam will bend to an arc of a circle of radius $R = \frac{EI}{M}$.

The central deflection,

$$\delta = \frac{l^2}{8R} = \frac{Ml^2}{8EI} \quad \dots\dots (5)$$

Also $M = fZ = \frac{2fI}{d} \quad \dots\dots (6)$

Substituting for M in (5),

$$\delta = \frac{l^2}{8EI} \times \frac{2f.I}{d} = \frac{1}{4} \times \frac{f.l^2}{E.d}$$

or $\frac{\delta}{l} \times \frac{d}{l} = \frac{1}{4} \times \frac{f}{E} \quad \dots\dots (C)$

(iv) Beam of 'Uniform strength' and constant depth d. Let f be the skin stress throughout. This beam also bends to an arc of a circle of radius R given by,

$$\frac{f}{d/2} = \frac{E}{R} \text{ or } R = \frac{E.d}{2f}$$

The central deflection δ is given by,

$$\delta = \frac{l^2}{8R} = \frac{fl^2}{4E.d}$$

In this case too,

$$\frac{\delta}{l} \times \frac{d}{l} = \frac{1}{4} \times \frac{f}{E} \quad \dots\dots (D)$$

Example 5.

A beam of uniform section is freely supported at the ends and carries a uniformly distributed load. Find the ratio of depth to span so that the maximum bending stress is 8 tons/in². when the central deflection is $\frac{1}{480}$ of the span. $E = 12800$ tons/in².

In this case,

$$\frac{\delta}{l} \times \frac{d}{l} = \frac{5}{24} \times \frac{f}{E}$$

$$\therefore \frac{1}{480} \times \frac{d}{l} = \frac{5}{24} \times \frac{8}{12800}$$

$$\therefore \quad \frac{d}{l} = \frac{1}{16}.$$

Example 6.

A timber beam carries a uniformly distributed load of 550 lb. per foot run over a span of 20 feet. The ends of the beam are freely supported. Determine the section of the beam if the central deflection is limited to $\frac{1}{2}''$ *and the maximum bending stress to 1000 lb/in*^{2}. $E = 1{\cdot}5 \times 10^6$ *lb/in*^{2}.

In the above case, if the depth of the beam is limited to 15″, calculate the necessary width.

If both the limits of stress and deflection are to be reached simultaneously, the depth d of the beam will be given by,

$$\frac{\delta}{l} \times \frac{d}{l} = \frac{5}{24} \times \frac{f}{E}$$

$$\frac{1}{2 \times 240} \times \frac{d}{240} = \frac{5}{24} \times \frac{1000}{1{\cdot}5 \times 10^6}$$

$$\therefore \qquad d = 16''.$$

To obtain the width b, the maximum B.M. is $\frac{Wl}{8}$ $= \frac{11000 \times 20}{8} \times 12 = 330000$ lb-inches. This is to be resisted at a stress of 1000 lb/in^2.

Since $M = f.Z$,

$$330000 = 1000 \times \tfrac{1}{6} \times b \times 16 \times 16$$

$$\therefore \qquad b = 7{\cdot}73'' \text{ say, } 8''.$$

A section 8″ wide and 16″ deep will do.

If the depth of section is limited to 15″, the permissible limits of stress and deflection cannot be reached at the same time. When the actual deflection at the centre is $\frac{1}{2}''$, the actual skin stress f will be less than 1000 lb/in^2. and will be given by,

$$\frac{\delta}{l} \times \frac{d}{l} = \frac{5}{24} \times \frac{f}{E} \text{ on substituting } \delta = \tfrac{1}{2}'' \text{ and } d = 15''$$

$$\therefore \qquad \frac{1}{2 \times 240} \times \frac{15}{240} = \frac{5}{24} \times \frac{f}{1{\cdot}5 \times 10^6}$$

$$\text{or} \qquad f = \frac{15000}{16} = 937{\cdot}5 \text{ lb/in}^2.$$

Equating the bending moment to the moment of resistance at this stress,

$$330000 = f \times \tfrac{1}{6} bd^2 = \frac{15000}{16} \times \tfrac{1}{6} \times b \times 15 \times 15$$

$$\therefore \qquad b = 9{\cdot}4'' \text{ say, } 9\tfrac{1}{2}''.$$

If the beam had been designed with a depth greater than 16″, the skin stress would reach the permissible limit of 1000 lb/in^2., while the deflection would be less than the given limit of $\frac{1}{2}''$.

9. Propped beams. Rigid and elastic props: In a simply supported beam of span *l* carrying a uniformly distributed load of *w* per foot run, if a central prop is introduced at the same level as the end supports, the reaction *P* of the prop can be obtained by consideration of deflection at the centre.

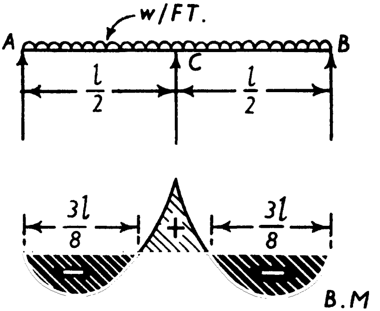


FIG. 123

If the prop is removed, the central deflection will be

$$y_C = \frac{5}{384} \cdot \frac{wl^4}{EI}.$$

If the prop is placed at the same level as the ends, the upward deflection due to *P*, viz., $\dfrac{Pl^3}{48EI}$ must neutralise the downward deflection due to the load.

$$\therefore \quad \frac{Pl^3}{48EI} = \frac{5}{384} \cdot \frac{wl^4}{EI}$$

$\therefore \quad P = \frac{5}{8}\,wl = \frac{5W}{8}$, where W is the total load on the beam. The end reactions will therefore be $\frac{3W}{16}$ each. The B.M. diagram for the beam is shown in fig. 123.

For slope and deflection, take the origin at A and x positive to the right. For the portion AC, consider a section X distant x from the origin.

Then, $M_x = -\frac{3wl}{16}.x + \frac{wx^2}{2}$

This is zero at $x = 0$ and at $x = \frac{3l}{8}$, the latter being a point of contraflexure. The maximum negative moment will be at $x = \frac{3l}{16}$ and its magnitude is,

$$M_{max} = -\frac{3wl}{16} \times \frac{3l}{16} + \frac{w}{2}\left(\frac{3l}{16}\right)^2 = -\frac{9}{512}wl^2.$$

The maximum positive moment will be at C and is given by, $M_{max} = -\frac{3wl}{16} \times \frac{l}{2} + \frac{w}{2}.\left(\frac{l}{2}\right)^2 = +\frac{wl^2}{32}$.

For slope and deflection,

$$EI\frac{d^2y}{dx^2} = -\frac{3wl}{16}.x + \frac{wx^2}{2}$$

$$\therefore \quad EI\frac{dy}{dx} = -\frac{3wlx^2}{32} + \frac{wx^3}{6} + C_1$$

At C, $\quad x = \frac{l}{2}$ and $\frac{dy}{dx} = 0$

$$\therefore \quad 0 = -\frac{3wl^3}{128} + \frac{wl^3}{48} + C_1$$

$$\therefore \quad C_1 = \frac{wl^3}{384}$$

$$\therefore \quad EI\frac{dy}{dx} = -\frac{3wlx^2}{32} + \frac{wx^3}{6} + \frac{wl^3}{384}.$$

Integrating,

$$EIy = -\frac{wlx^3}{32} + \frac{wx^4}{24} + \frac{wl^3}{384}.x + C_2$$

Since at A, $x = 0$ and $y = 0$, $C_2 = 0$.

$$\therefore \quad EIy = -\frac{wlx^3}{32} + \frac{wx^4}{24} + \frac{wl^3}{384}.x.$$

For the position of maximum deflection, $\frac{dy}{dx} = 0$

$$\therefore \quad -\frac{3wlx^2}{32} + \frac{wx^3}{6} + \frac{wl^3}{384} = 0$$

or $\quad 64x^3 - 36l.x^2 + l^3 = 0$

Solving, we get, $x = {\cdot}27l$

$$\therefore \quad EIy_{max} = wl^4\left\{-\frac{{\cdot}01968}{32} + \frac{{\cdot}0729^2}{24} + \frac{{\cdot}27}{384}\right\}$$

$$= wl^4\,[-{\cdot}0006152 + {\cdot}0002214 + {\cdot}0007]$$

$$= wl^4 \times {\cdot}0003062$$

$$\therefore \quad y_{max} = \frac{{\cdot}0003062wl^4}{EI}.$$

Elastic prop:

If the elasticity of the props is to be taken into account, the magnitudes of the reactions will alter. Let σ be the stiffness of each prop being measured by the force required to depress it by unity. If P be the reaction of the middle prop, the reactions of the end props will be $\frac{(wl - P)}{2}$ each. The end props will be depressed by $\frac{(wl - P)}{2\sigma}$ and the middle prop by $\frac{P}{\sigma}$. The difference of levels of the central prop and end props will therefore be,

$$\delta = \frac{P}{\sigma} - \frac{(wl - P)}{2\sigma} = \frac{(3P - wl)}{2\sigma}.$$

The upward deflection due to P has now to neutralise, not $\frac{5}{384}.\frac{wl^4}{EI}$ but $\left(\frac{5}{384}.\frac{wl^4}{EI} - \delta\right)$ only.

Equating,

$$\frac{Pl^3}{48EI} = \frac{5}{384}.\frac{wl^4}{EI} - \frac{(3P - wl)}{2\sigma}$$

$$\therefore \quad P = \tfrac{5}{8}.wl - \frac{24EI}{\sigma l^3}(3P - wl)$$

$$\therefore \quad P\left(1 + \frac{72EI}{\sigma l^3}\right) = wl\left\{\tfrac{5}{8} + \frac{24EI}{\sigma l^3}\right\}$$

or
$$P = wl \times \frac{\left(\frac{5}{8} + \frac{24EI}{\sigma l^3}\right)}{\left(1 + \frac{72EI}{\sigma l^3}\right)}$$

If the props have different stiffnesses, the magnitudes of reactions can be similarly derived.

10. Simply supported beam with an eccentric load *W*: Let the load W be placed at C, distant a from A and b from

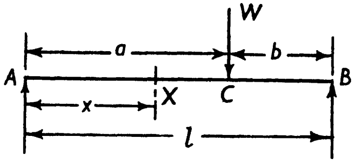


FIG. 124

B so that $a + b = l$. The reactions at the supports will be, $R_A = \frac{Wb}{l}$ and $R_B = \frac{Wa}{l}$. Now take the origin at A and consider a section X distant x from the origin, in the portion AC.

$$EI\frac{d^2y}{dx^2} = -R_A.x = -\frac{Wb}{l}.x$$

Integrating,

$$EI\frac{dy}{dx} = -\frac{Wbx^2}{2l} + C_1$$

At C, $x = a$ and $\frac{dy}{dx} = i_C \quad \therefore \; EIi_C = -\frac{Wba^2}{2l} + C_1.$

Substituting for C_1,

$$EI\frac{dy}{dx} = -\frac{Wbx^2}{2l} + \frac{Wba^2}{2l} + EIi_C \quad \ldots\ldots\ldots\ldots (1)$$

Integrating once more,

$$EIy = -\frac{Wbx^3}{6l} + \frac{Wba^2}{2l}.x + EIi_C\,x + 0 \ldots\ldots\ldots\ldots\ldots (2)$$

Since $y = 0$ at $x = 0$, the constant of integration is zero.

The expressions (1) and (2) which have been derived for slope and deflection at any section in the portion *AC*, will be useful as soon as i_C is evaluated.

Now consider a section *X* in the portion *CB*, at a distance x from the origin *A*.

$$EI\frac{d^2y}{dx^2} = -\frac{Wb}{l}.x + W\ (x-a).$$

Integrating,

$$EI\frac{dy}{dx} = -\frac{Wbx^2}{2l} + \frac{W\,(x-a)^2}{2} + C'_1$$

At $\qquad x = a,\ \dfrac{dy}{dx} = i_C$

$$\therefore \quad EIi_C = \frac{Wba^2}{2l} + C'_1$$

$$\therefore \quad C'_1 = \frac{Wba^2}{2l} + EIi_C.$$

Substituting,

$$EI\frac{dy}{dx} = -\frac{Wbx^2}{2l} + \frac{W\,(x-a)^2}{2} + \frac{Wba^2}{2l} + EIi_C \ldots\ldots\ldots (3)$$

Integrating once again,

$$EIy = -\frac{Wbx^3}{6l} + \frac{W\,(x-a)^3}{6} + \frac{Wba^2}{2l}.x + EIi_C.x + C'_2$$

At $\quad B,\ x = l$ and $y = 0$. Substituting,

$$0 = -\frac{Wbl^2}{6} + \frac{Wb^3}{6} + \frac{Wba^2}{2} + EIi_C.l + C'_2$$

$$\therefore \quad C'_2 = \frac{Wb}{6}(l^2 \quad b^2) - \frac{Wba^2}{2} - EIi_C l$$

$$\therefore \quad C'_2 = \frac{Wb\ (l+b)\ (l \quad b)}{6} - \frac{Wba^2}{2} \quad EIi_C l$$

$$= \frac{Wab\ (a+2b)}{6} \quad \frac{Wba^2}{2} - EIi_C.l$$

$$= -\frac{Wab\ (a-b)}{3} - EIi_C.l$$

$$\therefore \quad EIy = -\frac{Wbx^3}{6l} + \frac{W\ (x-a)^3}{6} + \frac{Wba^2}{2l}.x + EIi_C.x$$

$$-\frac{Wab\ (a-b)}{3} - EIi_C.l \ldots\ldots\ldots\ldots\ldots\ldots\ldots (4)$$

Expressions (3) and (4) give the slope and deflection at any section in the portion *CB*, provided i_C is evaluated.

The point *C* is common to both *AC* and *CB* and the deflection of *C* will be the same whether it is obtained from (2) or (4). Substituting $x = a$ in (2) and (4) and equating,

$$EIy_C = -\frac{Wba^3}{6l} + \frac{Wba^3}{2l} + EIi_C.a$$

$$= -\frac{Wba^3}{6l} + \frac{Wba^3}{2l} + EIi_C.a - \frac{Wab\ (a-b)}{3} - EIi_C.l$$

$$\therefore \quad EIi_C = -\frac{Wab\ (a-b)}{3l}$$

Substituting for EIi_C in (1) and (2), the expressions for slope and deflection for any section in the portion *AC* become,

$$EI\frac{dy}{dx} = -\frac{Wbx^2}{2l} + \frac{Wba^2}{2l} - \frac{Wab\ (a-b)}{3l}$$

$$= \frac{Wb\ (a^2 + 2ab - 3x^2)}{6l}$$

and $$EI\,y = -\frac{Wbx^3}{6l} + \frac{Wba^2}{2l}.x - \frac{Wab\ (a-b)}{3l}.x$$

$$= \frac{Wbx\{-x^2 + 3a^2 - 2a\ (a-b)\}}{6l} = \frac{Wbx(a^2 + 2ab - x^2)}{6l}$$

$$\therefore \quad y = \frac{Wbx\ (a^2 + 2ab - x^2)}{6EI.l} = \frac{Wbx\ (l^2 - b^2 - x^2)}{6EI.l}$$

For the position of maximum deflection,

$$\frac{dy}{dx} = 0 \text{ or } a^2 + 2ab - 3x^2 = 0$$

$$\therefore \quad 3x^2 = a^2 + 2ab \text{ and } x = \sqrt{\frac{a^2+2ab}{3}}$$

$$\therefore \quad y_{max} = \frac{Wb\,(a^2+2ab)^{\frac{3}{2}}}{9\sqrt{3}.EI.l}.$$

It may be noted that the maximum deflection occurs in the larger portion *AC* of the beam.

The deflection under the load is,

$$y_C = \frac{Wab \times 2ab}{6EI.l} = \frac{Wa^2b^2}{3EIl}, \text{ substituting } x = a.$$

Example 7.

A 12″ × 5″ × 32 lb. R.S.J. is simply supported over a span of 20 feet and carries a point-load of 5 tons at 4 feet from one support. Find the position and amount of maximum deflection and also the deflection at the centre. I_{xx} *for the section = 220 in*^{4}*.; E = 13000 tons/in*^{2}*.*

The maximum deflection will occur in the larger segment which we shall call a, so that $a = 16$ feet; $b = 4$ feet; $W = 5$ tons.

The deflection at a section distant x from the support is given by,

$$y = \frac{Wbx\,(a^2+2ab-x^2)}{6EI.l} = \frac{5 \times 4 \times x\,(256+128-x^2)}{6EI \times 20}$$

$$= \frac{x(384-x^2)}{6EI}.$$

For a maximum, $\frac{dy}{dx} = 0$

$$\therefore \quad 384 - 3x^2 = 0 \quad \text{or } x^2 = 128$$

and $x = 8\sqrt{2}$ feet $= 11{\cdot}31$ feet.

$$y_{max} = \frac{11{\cdot}31 \times (384-128)}{6EI}$$

$$= \frac{482{\cdot}56}{EI} = \frac{482{\cdot}56 \times 1728}{13000 \times 220} = {\cdot}2916''.$$

For the deflection at the centre $x = 10$.

Substituting in the general expression for y,

$$y_{10} = \frac{10\,(384-100)}{6EI} = \frac{473{\cdot}33}{EI} = \frac{473{\cdot}33 \times 1728}{13000 \times 220} = {\cdot}2861''.$$

Example 8.

A 12″ × 5″ × 32 lb. R.S.J. is freely supported over a span of 20 feet. It carries two concentrated loads of 3 and 4 tons at 4 and 15 feet from the left hand support. Find the position and amount of maximum deflection. $I_{xx} = 220$ *in*^{4}.; $E = 13000$ *tons/in*^{2}.

The maximum deflection will occur in the portion DE (fig. 125). We shall consider a section in this portion and put down its deflection due to the loads of 3 tons and 4 tons separately. The actual deflection will be the sum of the two. Taking the 4 ton-load first, let the origin be at A and consider a section X, distant x from A.

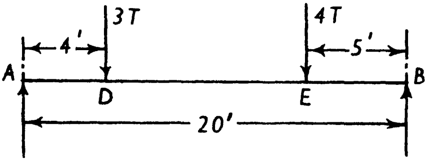


Fig. 125

$$a = 15';\ b = 5';\ W = 4 \text{ tons.}$$

$$y_1 = \frac{Wbx\,(a^2+2ab-x^2)}{6EI.l} = \frac{4 \times 5x\,(225+150-x^2)}{6EI \times 20}$$

$$= \frac{x\,(375-x^2)}{6EI}$$

Next, consider the 3 ton-load and take the origin at B.

Let x' be the distance of the section X from B so that $x' = 20 - x$.

Now, $a' = 16$ feet; $b' = 4$ feet; $W' = 3$ tons.

$$y_2 = \frac{3 \times 4 \times x'\,(256+128-x'^2)}{6EI \times 20} = \frac{x'\,(384-x'^2)}{10EI}$$

$$\text{Total deflection } y = y_1 + y_2 = \frac{x\,(375-x^2)}{6EI} + \frac{x'\,(384-x'^2)}{10EI}$$

$\therefore\ 30EIy = 5x\,(375-x^2) + 3x'\,(384-x'^2)$,

where $x' = 20 - x$. For a maximum, differentiate with regard to x and equate to zero.

$$\therefore \quad 30EI\frac{dy}{dx} = 5 \times 375 - 15x^2 - 3 \times 384 + 9x'^2 = 0,$$

since $dx' = -dx$.

$$\therefore \quad 625 - 5x^2 - 384 + 3\,(20 - x)^2 = 0 \text{ or, } 2x^2 + 120x = 1441$$

$$\therefore \quad x^2 + 60x = 720{\cdot}5 \quad \therefore\ (x + 30)^2 = 1620{\cdot}5$$

$$\therefore \quad x + 30 = 40{\cdot}27$$

and $x = 10{\cdot}27$ feet. $x' = 9{\cdot}73$ feet.

$$y_{max} = \frac{10{\cdot}27\ (375 - 105{\cdot}47)}{6EI} + \frac{9{\cdot}73\ (384 - 94{\cdot}7)}{10EI}$$

$$= \frac{461{\cdot}3}{EI} + \frac{281{\cdot}5}{EI} = \frac{742{\cdot}8}{EI} = \frac{742{\cdot}8 \times 1728}{13000 \times 220} = {\cdot}4488''.$$

11. Macaulay's method*: Problems in deflections of beams, particularly where point loads are concerned, become delightfully simple by a method which will now be described. It is due to Mr. W. H. Macaulay and essentially consists in the manner of the successive integrations of the expression for bending moment so that the same constants of integration are valid for all portions of the beam even though the law of the bending moment varies from portion to portion. The method is best understood by a simple example. Let us reconsider the case of a single eccentric load *W* on a simply supported beam of span *l*.

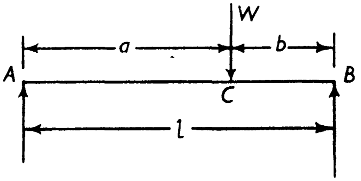


FIG. 126

The reactions of the supports are,

$$R_A = \frac{Wb}{l} \text{ and } R_B = \frac{Wa}{l}.$$

Take the origin at *A*. The law of the bending moment at any section in *AC*, distant *x* from *A* is,

$$M_x = -\frac{Wb}{l}.x, \quad x \text{ being} < a.$$

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* This method is originally due to A. Clebsch.

The law of the bending moment at a section in *CB*, distant x from A is,

$$M_x = -\frac{Wbx}{l} + W(x-a), \ x \text{ being} > a.$$

We can make one statement of the law of the bending moment for all sections of the beam from the end A to B as under:

$$M_x = EI\frac{d^2y}{dx^2} = -\frac{Wb}{l}.x \ \vdots \ + W(x-a).$$

For sections in *AC* where x is $< a$, stop at the dotted line. For sections in *CB* $(x > a)$, add the expression beyond the dotted line.

Now integrate for slope,

$$\therefore \ EI\frac{dy}{dx} = -\frac{Wbx^2}{2l} + C_1 \vdots + \frac{W(x-a)^2}{2} \dots\dots\dots\dots\dots\dots (1)$$

It is particularly to be noted that the integration of $(x-a)$ has been made for the expression as a whole and not for individual terms of the expression. This is just what makes the Macaulay method effective because, now C the constant of integration is not only valid for slopes at sections in the portion *AC* but also for those in the portion *CB*.

Consider the point C which is common to both *AC* and *CB*. As a point in *AC*, the slope i_C given by,

$EIi_C = -\dfrac{Wba^2}{2l} + C_1$, substituting $x = a$ and stopping at the dotted line.

As a point in *CB*, the slope at C will be obtained by substituting $x = a$ in the whole expression

$EI\dfrac{dy}{dx} = -\dfrac{Wbx^2}{2l} + C_1 + \dfrac{W(x-a)^2}{2}$. And we get,

$EIi_C = -\dfrac{Wba^2}{2l} + C_1$, as before, since the last expression is zero when we put $x = a$. The constant C_1 is therefore valid both for *AC* and *CB*.

Integrate the expression (1) for deflections.

$$\therefore\quad \underline{EIy = -\frac{Wbx^3}{6l} + C_1.x + C_2} + \frac{W(x-a)^3}{6} \ldots\ldots\ldots\ldots(2)$$

Again, the expression $(x-a)^2$ has been integrated as a whole so that the constant of integration C_2 is valid for deflections in AC as well as in CB.

We have thus worked out expressions (1) and (2) for slopes and deflections for all sections of the beam from one end to the other. For sections in AC, we stop at the dotted line; for section in CB, we add the expression beyond the dotted line.

We have, now, only to evaluate the constants C_1 and C_2, which is easily done because we know two positions where y is zero, viz., at A where $x = 0$ and B, where $x = l$.

Remembering that A lies in the portion AC and putting $x = 0$ and $y = 0$ in the expression $EIy = -\frac{Wbx^3}{6l} + C_1x + C_2$, we get $C_2 = 0$.

Again, since B lies in the portion CB, putting $x = l$ and $y = 0$ in the expression, $EIy = -\frac{Wbx^3}{6l} + C_1x + C_2 + \frac{W(x-a)^3}{6}$, we get,

$$0 = -\frac{Wbl^2}{6} + C_1.l + \frac{W(l-a)^3}{6},\ \text{since } C_2 = 0$$

$$\text{or}\quad 0 = C_1.l - \frac{Wb}{6}(l^2 - b^2),\ \text{since } l - a = b$$

$$\therefore\quad C_1 = \frac{Wb}{6l}(l+b)(l-b) = \frac{Wab\ (a+2b)}{6l}.$$

The expressions for slope and deflection for all sections from A to B, are therefore,

$$EI\frac{dy}{dx} = -\frac{Wbx^2}{2l} + \frac{Wab\ (a+2b)}{6l} + \frac{W(x-a)^2}{2}$$

$$\text{and}\quad EIy = -\frac{Wbx^3}{6l} + \frac{Wab\ (a+2b)}{6l}x + \frac{W\ (x-a)^3}{6}.$$

It will be noticed that the constant C_2 is usually zero, since at $x = 0$, $y = 0$. The work therefore reduces to evaluating only the constant C_1 which is done by putting $y = 0$ at $x = l$ in the *whole* expression for deflection.

For any section in the portion AC ($x < a$),

$$EI\frac{dy}{dx} = -\frac{Wbx^2}{2l} + \frac{Wab\ (a + 2b)}{6l} = \frac{Wb}{6l}(a^2 + 2ab - 3x^2)$$

and $$EIy = -\frac{Wbx^3}{6l} + \frac{Wab\ (a + 2b)}{6l}.x = \frac{Wbx(a^2 + 2ab - x^2)}{6l},$$

as already obtained laboriously in art 10.

We shall now apply the method to examples 7 and 8 above so that the simplicity of the method becomes evident.

Example 7a.

Referring to example 7, the reactions at the supports are $R_A = 1$ ton and $R_B = 4$ tons; $a = 16$ feet; $b = 4$ feet; $W = 5$ tons in fig. 126.

Taking the origin at A, the law of the bending moment for all sections from A to B is,

$$EI\frac{d^2y}{dx^2} = -1.x \quad + 5\ (x - 16)$$

Integrating,

$$EI\frac{dy}{dx} = -\frac{x^2}{2} + C_1 \quad + \tfrac{5}{2}\ (x - 16)^2$$

Integrating once again,

$$EIy = -\frac{x^3}{6} + C_1.x + C_2 \quad + \tfrac{5}{6}\ (x - 16)^3$$

Since $y = 0$ at $x = 0$, $C_2 = 0$.

To evaluate C_1, put $y = 0$ at $x = 20'$ in the whole expression.

$\therefore\quad 0 = -\tfrac{1}{6} \times 8000 + 20C_1 + \tfrac{5}{6} \times 64$

$\therefore\quad C_1 = 64.$

The expressions for slope and deflection are,

$$EI\frac{dy}{dx} = -\frac{x^2}{2} + 64 \quad + \tfrac{5}{2}\ (x - 16)^2$$

$$EIy = -\frac{x^3}{6} + 64x \quad + \tfrac{5}{6}\ (x - 16)^3.$$

The maximum deflection occurs in AC where, $\frac{dy}{dx} = 0$.

$$\therefore \quad -\frac{x^2}{2} + 64 = 0$$

$$\text{or} \quad x^2 = 128$$

$$\text{and} \quad x = 8\sqrt{2} = 11{\cdot}31 \text{ feet.}$$

The maximum deflection is given by,

$$EIy_{max} = x\ (64 - \frac{x^2}{6}),\ x \text{ being } 11{\cdot}31'$$

$$\therefore \quad EIy_{max} = 11{\cdot}31\ (64 - \frac{128}{6}) = \frac{11{\cdot}31 \times 128}{3} = 482{\cdot}56$$

$$\therefore \quad y_{max} = \frac{482{\cdot}56}{EI} = \frac{482{\cdot}56 \times 1728}{13000 \times 220} = {\cdot}2916'' \text{ as before.}$$

The deflection at the centre is obtained by putting $x = 10$ in the expression $EIy = -\frac{x^3}{6} + 64x$.

$$\therefore \quad EIy_{10} = -\frac{1000}{6} + 640 = \frac{2840}{6} = 473{\cdot}33$$

$$\therefore \quad y_{10} = \frac{473{\cdot}33}{EI} = \frac{473{\cdot}33 \times 1728}{13000 \times 220} = {\cdot}2861''.$$

Example 8a.

Referring to fig. 125, the reactions at the supports will be,

$$R_A = \frac{4 \times 5 + 3 \times 16}{20} = 3{\cdot}4 \text{ tons.}$$

$$R_B = 3{\cdot}6 \text{ tons.}$$

Taking the origin at A, the law of the bending moment for all sections for A to B is,

$$EI\frac{d^2y}{dx^2} = -3{\cdot}4x \quad + 3\ (x-4) \quad + 4\ (x-15)$$

Integrating,

$$EI\frac{dy}{dx} = -1{\cdot}7x^2 + C_1 \quad + \tfrac{3}{2}\ (x-4)^2 \quad + 2(x-15)^2$$

Integrating once more,

$$EIy = -\frac{1{\cdot}7}{3}x^3 + C_1x + 0 \quad + \tfrac{1}{2}\ (x-4)^3 \ + \tfrac{2}{3}\ (x-15)^3$$

To evaluate C_1, put $y = 0$ at $x = 20$, in the whole expression for y.

$$0 = -\frac{1\cdot 7}{3} \times 8000 + 20C_1 + \tfrac{1}{2} \times 16^3 + \tfrac{2}{3} \times 125$$

$$\therefore \quad 20C_1 = 2402$$

$$\therefore \quad C_1 = 120\cdot 1.$$

The slope and deflection relations are, therefore,

$$EI\frac{dy}{dx} = -1\cdot 7x^2 + 120\cdot 1 \quad + \tfrac{3}{2}(x-4)^2 \;+ 2(x-15)^2$$

and $$EIy = -1\cdot 7\frac{x^3}{3} + 120\cdot 1x + \tfrac{1}{2}(x-4)^3 \;+ \tfrac{2}{3}(x-15)^3.$$

The maximum deflection occurs in the middle portion, between $x = 4$ and $x = 15$. Equating $\frac{dy}{dx}$ to zero. We get,

$$0 = -1\cdot 7x^2 + 120\cdot 1 + \tfrac{3}{2}(x-4)^2$$

$$\therefore \quad -3\cdot 4x^2 + 240\cdot 2 + 3x^2 - 24x + 48 = 0$$

or $$\cdot 4x^2 + 24x = 288\cdot 2$$

$$\therefore \quad x^2 + 60x = 720\cdot 5$$

$$(x+30)^2 = 1620\cdot 5$$

$$x + 30 = 40\cdot 27$$

$$x = 10\cdot 27 \text{ feet.}$$

Substituting

$x = 10\cdot 27$ in the expression for deflection,

$$EIy_{max} = -\frac{1\cdot 7}{3}(10\cdot 27)^3 + 120\cdot 1 \times 10\cdot 27 + \tfrac{1}{2}(6\cdot 27)^3$$

$$= -613\cdot 8 + 1233\cdot 4 + 123\cdot 2$$

$$= 742\cdot 8$$

$$\therefore \quad y_{max} = \frac{742\cdot 8}{EI} = \frac{742\cdot 8 \times 1728}{13000 \times 220} = \cdot 4488''.$$

12. Variable loading on a beam of uniform section: If the beam carries a system which varies from section to section, the slope and deflection at a section can be worked out, as usual, by successive integration of the relation $EI\frac{d^2y}{dx^2} = M$, provided M is an integrable function.

There is a more mechanical way in which, by repeated integration of the function for the rate of loading at a cross-section, we successively derive the S.F., B.M., slope and deflection at the section. The relations between the rate of loading w and F and M at a section are:

$$\frac{dF}{dx} = w \text{ and } \frac{dM}{dx} = F.$$

Since $M = EI\dfrac{d^2y}{dx^2}$, $\dfrac{dM}{dx} = F = EI\dfrac{d^3y}{dx^3}$ and

$$\frac{d^2M}{dx^2} = \frac{dF}{dx} = EI\frac{d^4y}{dx^4} = w.$$

Taking the basic relation $EI\dfrac{d^4y}{dx^4} = w$, successive integration will yield $EI\dfrac{d^3y}{dx^3}$ or F, $EI\dfrac{d^2y}{dx^2}$ or M, $EI\dfrac{dy}{dx}$ and EIy.

Example 9.

An 18″ × 7″ × 75 lb. R.S.B. is simply supported over a span of 18 feet and carries a distributed load which varies uniformly from 1 ton per foot run at one end to 3 tons per foot run at the other. Find the position and amount of maximum deflection. For the R.S.B., $I_{xx} = 1149$ *in*^{4}.; $E = 13000$ *tons/in*^{2}.

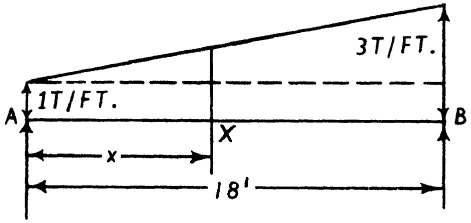


FIG. 127

The load may be divided into: (i) a uniform load of 1 ton per foot run throughout and (ii) a triangular load which is zero at A, increasing to 2 tons/ft. at B. The reactions to (i) are 9 tons at each end. The reactions to (ii) are $\frac{1}{3} \times 18 = 6$ tons at A and $\frac{2}{3} \times 18 = 12$ tons at B. The total reactions are, therefore, $R_A = 9 + 6 = 15$ tons and $R_B = 9 + 12 = 21$ tons.

Now take the origin at A and consider a section X at a distance of x from the origin.

$$EI\frac{d^2y}{dx^2} = M = -15x + x.\frac{x}{2} + \tfrac{1}{2} \times \frac{2x}{18} \times x \times \frac{x}{3}$$

$$= -15x + \frac{x^2}{2} + \frac{x^3}{54}.$$

This expression could have been derived by considering the rate of loading at X which is $w = \left(1 + \frac{2x}{18}\right)$ tons per foot run.

$$\therefore \qquad EI\frac{d^4y}{dx^4} = \left(1 + \frac{x}{9}\right)$$

Integrating, $EI\frac{d^3y}{dx^3} = x + \frac{x^2}{18} + F_A$.

Integrating once more,

$EI\frac{d^2y}{dx^2} = \frac{x^2}{2} + \frac{x^3}{54} + F_Ax + 0$, since $EI\frac{d^2y}{dx^2}$ or M is zero at $x = 0$.

It is also zero at $x = 18$.

$$\therefore \qquad 0 = 162 + \frac{18^3}{54} + 18F_A$$

$$\therefore \qquad F_A = -15 \text{ tons}$$

$\therefore \qquad EI\frac{d^2y}{dx^2} = -15x + \frac{x^2}{2} + \frac{x^3}{54}$, as already obtained from first principles.

Continuing the integrations,

$$EI\frac{dy}{dx} = -\frac{15x^2}{2} + \frac{x^3}{6} + \frac{x^4}{216} + C_1$$

and $EIy = -\frac{5x^3}{2} + \frac{x^4}{24} + \frac{x^5}{1080} + C_1x + 0$,

since $y = 0$ at $x = 0$. Also at $x = 18$, $y = 0$

$$\therefore \qquad 0 = -810 + 243 + 97{\cdot}2 + C_1$$

$$\therefore \qquad C_1 = 469{\cdot}8.$$

The laws of slope and deflection, therefore, are,

$$EI\frac{dy}{dx} = -\frac{15x^2}{2} + \frac{x^3}{6} + \frac{x^4}{216} + 469{\cdot}8$$

$$EIy = -\frac{5x^3}{2} + \frac{x^4}{24} + \frac{x^5}{1080} + 469{\cdot}8x.$$

For maximum deflection, $\frac{dy}{dx} = 0$

$$\therefore \quad -\frac{15x^2}{2} + \frac{x^3}{6} + \frac{x^4}{216} + 469{\cdot}8 = 0.$$

Solve for x by trial and error. Trying $x = 9$, the expression reduces to

$-607{\cdot}5 + 121{\cdot}5 + 30{\cdot}375 + 469{\cdot}8$

or $+14{\cdot}175$, instead of zero.

A little higher value of x will satisfy the equation.

Try $x = 9{\cdot}18$

Then, $x^2 = 84{\cdot}26$

$x^3 = 773{\cdot}4$

$x^4 = 7099$

$x^5 = 65160.$

Substituting in the equation for $EI\frac{dy}{dx}$, the R.H. side reduces to

$$\frac{15}{2} \times 84{\cdot}26 + \frac{773{\cdot}4}{6} + \frac{7099}{216} + 469{\cdot}8$$

or $-632 + 128{\cdot}9 + 32{\cdot}86 + 469{\cdot}8$

or $-0{\cdot}44$, which is near enough to zero.

$$\therefore \quad EIy_{max} = -\frac{5}{2} \times 773{\cdot}4 + \frac{1}{24} \times 7099 + \frac{65160}{1080} + 469{\cdot}8 \times 9{\cdot}18$$

$$= -1933{\cdot}5 + 295{\cdot}8 + 60{\cdot}34 + 4312$$

$$= 2734{\cdot}64$$

$$\therefore \quad y_{max} = \frac{2734{\cdot}64 \times 1728}{13000 \times 1149} = {\cdot}3163''.$$

13. Slope and deflection by the use of the B. M. diagram: The area of the B.M. diagram and its moment can be utilised to evaluate the slope and deflection at any section of a beam.

Let M be the bending moment at a section X of beam at a distance of x from A (fig 128). With the origin at A, $EI\frac{d^2y}{dx^2} = M.$

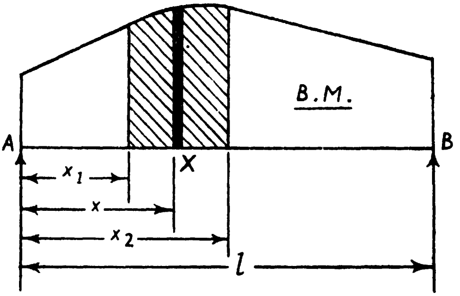


FIG. 128

Integrating between x_1 and x_2, we get,

$$EI\int_{x_1}^{x_2}\frac{d^2y}{dx^2}.dx = \int_{x_1}^{x_2} M.dx$$

$\therefore$ $EI\left\{\frac{dy}{dx}\right\}_{x_1}^{x_2}$ =area of the B.M. diagram between x_1 and x_2

$\therefore$ $EI\ (i_2 - i_1) =$ „ „ „ „ „ „ „ „

$\therefore$ The change of slope between two sections of a beam

$$= \frac{\text{area of the B.M. diagram between the sections}}{EI} \quad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Similarly, multiply both sides of the equation $EI\frac{d^2y}{dx^2} = M$ by x and then integrate between x_1 and x_2.

$$\therefore \quad EI\int_{x_1}^{x_2} x\frac{d^2y}{dx^2}dx = \int_{x_1}^{x_2} M.x.dx$$

$\therefore$ $EI\left\{x\frac{dy}{dx} - y\right\}_{x_1}^{x_2}$ = moment of the B.M. diagram between x_1 and x_2 about A.

or $EI\left\{(x_2i_2 - y_2) - (x_1i_1 - y_1)\right\}$ = „ „ „ „ „ .... (2)

These two relations are enough to obtain the slope and deflection in simple cases.

Consider a cantilever of span l carrying a point load W at the free end. We know that the slope and deflection at the free end are $\frac{Wl^2}{2EI}$ and $\frac{Wl^3}{3EI}$ respectively. We can obtain them easily by the use of the B.M. diagram which is a triangle. The area of the B.M. diagram between A and B is $\frac{1}{2}\,Wl.l = \frac{Wl^2}{2}$. The distance of its c.g. from A is $\frac{l}{3}$ and from B it is $\frac{2l}{3}$.

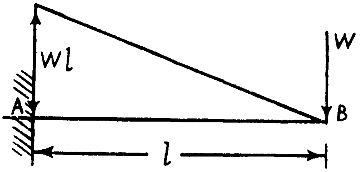


FIG. 129

Now take the origin at B and x positive to the left.

From equation (1), we get,

$EI(i_A - i_B)$ = area of the B.M. diagram between B and A $= \frac{Wl^2}{2}$.

But $i_A = 0$

$\therefore\quad i_B = -\frac{Wl^2}{2EI}$, the negative sign being due to our change of direction for x.

Using the relation (2) we get,

$EI\ [(li_A - y_A) - (0.i_B - y_B)]$ = moment of the B.M. diagram about $B = \frac{Wl^2}{2}.\frac{2l}{3} = \frac{Wl^3}{3}$.

Since i_A and y_A are both zero, we get,

$$EIy_B = \frac{Wl^3}{3}$$

And $$y_B = \frac{Wl^3}{3EI}$$

We can similarly obtain the slope and deflection at the free end of a cantilever carrying a uniformly distributed load *w* over its span *l*.

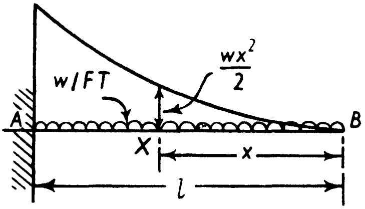


FIG. 130

Take the origin at *B* and *x* positive to the left. The B.M. at a distance *x* from the origin is,

$$EI\frac{d^2y}{dx^2} = +\frac{wx^2}{2}.$$

Integrating over the whole range,

$EI\,(i_A - i_B)$ = area of the B.M. diagram between *B* and *A*

$$= \int_0^l \frac{wx^2}{2}\,dx = \frac{wl^3}{6}.$$

Since $$i_A = 0,\ i_B = -\frac{wl^3}{6EI}.$$

Also $EI[(li_A - y_A) - (0.i_B - y_B)]$ = moment of the B.M. diagram about the origin *B*.

Since i_A and y_A are both zero,

$$EIy_B = \int_0^l \frac{wx^2}{2}.x.dx = \int_0^l \frac{wx^3}{2}\,dx = \frac{wl^4}{8}$$

$$\therefore\quad y_B = \frac{wl^4}{8EI},\text{ as before.}$$

We shall, however, derive regular expressions for the slope *i* and deflection *y* at any section of a beam in terms of the area of the B.M. diagram and its moment.

14. Beam supported at the ends: Take the origin at A and x positive to the right. At any section X, distant x from A,

$$EI\frac{d^2y}{dx^2} = M.$$

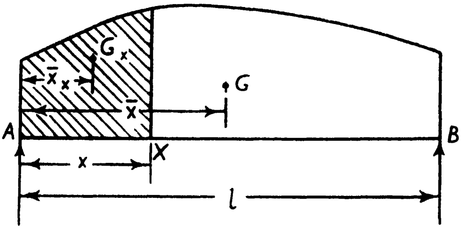


FIG. 131

Integrating between 0 to x,

$$EI\,(i_x - i_A) = \int_0^x M.dx = a_x$$ where a_x is the area of the B.M. diagram from A to X.

$$\therefore\ \ i_x = i_A + \frac{a_x}{EI} \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \ (1)$$

This relation will give the slope at any section as soon as we evaluate i_A, slope at the origin. Multiply both sides of the relation $EI\frac{d^2y}{dx^2} = M$ by x and integrate over the whole range from A to B.

We have, $\int_0^l EIx\frac{d^2y}{dx^2}dx = \int_0^l M.x.dx.$

$\therefore\ \ EI\left\{x\frac{dy}{dx} - y\right\}_0^l$ = moment of the whole B.M. diagram about $A = a.\bar{x}$, where a is the area of the B.M. diagram and $\bar{x}$ is the distance of its c.g. from A.

Substituting the limits,

$$EI\,[(li_B - y_B) - (0.i_A \quad y_A)] = a.\bar{x}.$$

Since y_A and y_B are both zero if the supports are at the same level,

$$EIl.i_B = a\bar{x}$$

$$\text{or} \quad i_B = \frac{a\bar{x}}{EIl} \qquad \text{(2)}$$

This expression for EIi_B can be easily remembered because of its analogy with the expressions for the reaction at the supports of a beam for a given load system. To obtain the reactions we take moments about a support; thus, by taking moments about A we have,

$R_B \times l =$ moment of the load system about A.

To obtain the slope at the support B, we take moments of the B.M. diagram about A and obtain EIi_B just as we get R_B for the load system.

$$\text{Similarly, } i_A = -\frac{a\,(l-\bar{x})}{EI.l} \qquad \text{(3)}$$

In using these expressions, care will have to be taken about conventional signs. With the usual loading on a simply supported beam, the beam bends concave upwards and therefore, according to our convention, the bending moment M at a section is usually negative. The area a of the B.M. diagram will be negative and therefore i_B will be negative. Since i_A is positive, the minus sign is placed before the expression for i_A in (3).

The slope i_x at any section can now be obtained from (1), since i_A is known.

For deflection, multiplying both sides of the expression $EI\frac{d^2y}{dx^2} = M$ by x and integrating between 0 and x, we get,

$$EI\int_0^x \frac{xd^2y}{dx^2}dx = \int_0^x M.x.dx$$

$$\therefore \quad EI\left\{(x\frac{dy}{dx} - y)\right\}_0^x = \text{moment of the B.M. diagram}$$

between A and X, about the origin.

$\therefore \quad EI \;\{(xi_x - y) - (0.i_A - 0)\} = a_x\bar{x}_x$, where a_x is the area of the B.M. diagram from A to X and $\bar{x}_x$ is the distance of its c.g. from A.

$$\therefore \quad y = x.i_x - \frac{a_x\bar{x}_x}{EI}$$

Substituting for i_x from (1),

$$y = x\left(i_A + \frac{a_x}{EI}\right) - \frac{a_x\bar{x}_x}{EI}$$

$$= xi_A + \frac{a_x(x - \bar{x}_x)}{EI}$$

But $a_x\ (x - \bar{x}_x)$ is the moment of the area a_x about X. Therefore,

$$y = xi_A + \frac{\text{moment of } a_x \text{ about } X}{EI} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (4)$$

The relations (1) and (4) respectively give the values of slope and deflection at any section X in terms of the area and moment of the area of the B.M. diagram. It must be remembered that the area a_x for a simply supported beam with the usual loading is negative.

We shall now illustrate this method by applying it to the standard case of a simply supported beam of span l, carrying a concentrated load W (i) at the centre (ii) eccentric.

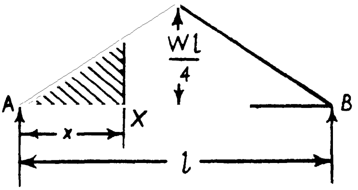


Fig. 132

(i) Central load W; span l.

At a section X distant x from A,

$$M = -\tfrac{1}{2}\,W.x.$$

The area $a_x = -\frac{x}{2}.\frac{Wx}{2} = -\frac{Wx^2}{4}$

The distance of its c.g. from $X = \frac{x}{3}$. To obtain i_A take moments of the entire B.M. diagram about B.

$$EIi_A \times l = \tfrac{1}{2} \times l \times \frac{Wl}{4} \times \frac{l}{2} = \frac{Wl^3}{16}$$

$$\therefore \quad i_A = \frac{Wl^2}{16EI} \quad \therefore \; i_x = i_A + \frac{a_x}{EI} = \frac{Wl^2}{16EI} - \frac{Wx^2}{4EI}$$

and $$y = xi_A + \frac{\text{moment of } a_x \text{ about } X}{E.I}$$

$$= \frac{Wl^2}{16EI}.x - \frac{Wx^3}{12EI}$$

These results were obtained by integration in art. 7 of this chapter.

(ii) Eccentric load W; span l.

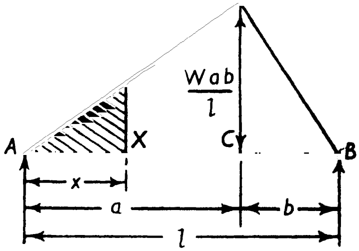


FIG. 133

The reactions at the supports are $\frac{Wb}{l}$ and $\frac{Wa}{l}$ respectively.

For a section X at a distance of x from A,

$$M = -\frac{Wb}{l}.x \text{ where } x < a.$$

The area $a_x = -\frac{x}{2}.\frac{Wbx}{l} = -\frac{Wbx^2}{2l}$

The distance of its c.g. from $X = \frac{x}{3}$.

To obtain the slope at A take moments of the whole B.M. diagram about B.

We have, $$EIi_A.l = \frac{Wab}{l}.\frac{b}{2}.\frac{2b}{3} + \frac{Wab}{l}.\frac{a}{2}\left(b + \frac{a}{3}\right)$$

$$EI.i_A l = \frac{Wab}{6l}\left\{2b^2 + a\ (3b + a)\right\}$$

$$= \frac{Wab}{6l}\,(a+b)\,(a+2b).$$

Since $(a+b) = l$, $i_A = \dfrac{Wab\,(a+2b)}{6EI.l}$

For any section in AC,

$$i_x = i_A + \frac{a_x}{EI} = + \frac{Wab\,(a+2b)}{6EIl} - \frac{Wbx^2}{2EIl}$$

$$= \frac{Wb\,(a^2+2ab-3x^2)}{6E.I.l}$$

and $y = xi_A + \dfrac{\text{moment of } a_x \text{ about } X}{EI}$

$$= \frac{Wab\,(a+2b)x}{6EIl} - \frac{Wbx^3}{6EIl} = \frac{Wbx\,(a^2+2ab-x^2)}{6E.I.l},$$

results which have been obtained before by other methods.

15. Impact loading on beams: If a load W is dropped on to the middle of a simply supported beam from a height h, the beam will suffer an instantaneous maximum deflection δ, vibrate and finally settle down to the position which it would have taken if the load W had been gradually applied. It is necessary to evaluate the maximum instantaneous deflection δ because in that position, the instantaneous stress induced is also a maximum and it must not exceed the permissible bending stress for the material. Such problems are best solved by working out the gradually applied load P which is equivalent to the impact load W. The work supplied to the beam and stored by it, in either case, is $W\,(h+\delta) = \frac{1}{2}P.\delta$, where δ is the maximum inst. deflection and P is the equivalent gradually applied load.

Also, $\delta = \dfrac{Pl^3}{48EI}$.

With the help of these two equations, solve for P and δ. The maximum instantaneous stress can now be easily obtained, since, $\dfrac{P.l}{4} = f.Z$.

Example 10.

An 18″ × 6″ × 55 lb. R.S.B. is freely supported over a span of 16 feet. A load of 2 tons is dropped on to the middle of the beam from a height of ½″. Find the maximum instantaneous deflection and stress induced in the beam, given, I_{xx} for the section = 841·76 in⁴., and E = 13000 tons/in².

$$W\,(h+\delta) = \frac{P}{2} \times \delta$$

$$\therefore\quad 2\,(\tfrac{1}{2}+\delta) = \frac{P}{2} \times \delta$$

$$\therefore\quad \delta\,(P-4) = 2.$$

Also, $\delta = \dfrac{Pl^3}{48EI}$

Substituting for δ,

$$P(P-4) = \frac{96EI}{l^3} = \frac{96 \times 13000 \times 841{\cdot}76}{16^3 \times 1728} = 148{\cdot}5$$

$$\therefore\quad (P-2)^2 = 152{\cdot}5$$

$$\therefore\quad P-2 = 12{\cdot}35$$

and $P = 14{\cdot}35$ tons.

$$M_{max} = \frac{P.l}{4} = \frac{14{\cdot}35 \times 16 \times 12}{4} = 688{\cdot}8 \text{ ton-inches.}$$

$$Z \text{ for the section} = \frac{841{\cdot}76}{9} = 93{\cdot}53 \text{ in}^3.$$

$$\therefore\quad f = \frac{M}{Z} = \frac{688{\cdot}8}{93{\cdot}53} = 7{\cdot}365 \text{ tons/in}^2.$$

The maximum instantaneous deflection δ may be obtained from $\delta = \dfrac{Pl^3}{48EI}$ or from $\dfrac{\delta}{l} \times \dfrac{d}{l} = \dfrac{1}{6} \times \dfrac{f}{E}$.

$$\therefore\quad \frac{\delta}{192} \times \frac{18}{192} = \frac{1}{6} \times \frac{7{\cdot}365}{13000}$$

$$\therefore\quad \delta = {\cdot}1933''.$$

16. Strain-energy of beams: Any system of external forces acting upon a structural member does work upon it and within the elastic limit, this work is stored by the member as strain-energy. The expression for the strain-energy in terms of the stress induced in the case of a member under axial pull or push is $\frac{f^2}{2E} \times$ volume of the member, where f is the uniform tensile or compressive stress induced in the section. In the case of transverse loading on beams, we have seen that the stresses induced at a section are tensile and compressive but they vary from zero at the neutral axis to a maximum at the extremities. Even the stresses at the extremities vary from section to section as the bending moment varies. If f is the maximum intensity of bending stress corresponding to the maximum bending moment on the beam, we can express the strain-energy of the beam as $c \times \frac{f^2}{E} \times$ volume of the beam, where c is a constant depending upon the cross-section of the beam and the nature of loading on it. We shall investigate the value of c in a few standard cases.

(i) Simply supported beam of uniform section: central load W, gradually applied.

The work done upon the beam by the gradually applied load = average force × displacement

$= \frac{1}{2} W \times \delta$ where δ is the central deflection.

The work supplied and stored $= \frac{1}{2} \times W \times \frac{Wl^3}{48EI} = \frac{W^2l^3}{96EI}$.

This is to be expressed in terms of the maximum stress f and the dimensions of the section.

$$\text{Now, maximum B.M.} = \frac{Wl}{4} = fZ = \frac{2fI}{d}$$

$$\therefore \qquad Wl = \frac{8fI}{d}.$$

Substituting for Wl,

$$\text{the strain-energy} = \frac{l}{96EI} \times \frac{64f^2.I^2}{d^2}$$

$$= \frac{2}{3}.\frac{f^2}{E} \times \frac{l.I}{d^2}$$

Putting $I = Ak^2$, the expression becomes,

$$\frac{2}{3} \times \frac{k^2}{d^2} \times \frac{f^2}{E} \times A \times l \text{ or } \frac{2}{3} . \frac{k^2}{d^2} . \frac{f^2}{E} \times \text{volume of the beam.}$$

The value of the constant c, in this case, is $\frac{2}{3} . \frac{k^2}{d^2}$.

If the beam section is rectangular of width b and depth d,

$$I = \tfrac{1}{12}\, bd^3 = bd.k^2$$

$$\therefore \quad k^2 = \tfrac{1}{12}\, d^2$$

i.e. $\frac{k^2}{d^2} = \frac{1}{12}$, and the value of c is therefore $\frac{1}{18}$.

For a beam of solid circular section of diameter d,

$$I = \frac{\pi}{64} d^4 = \frac{\pi}{4}\, d^2 \times k^2$$

$\therefore \ \frac{k^2}{d^2} = \frac{1}{16}$ and c is $\frac{1}{24}$.

(ii) Simply supported beam with a uniformly distributed load w.

A uniformly distributed load, or for that matter any loading can be looked upon as a series of point-loads acting on the beam. Consider an element δx of the beam at a distance x from the origin. The load on the element is $w.\delta x$. If y is the deflection, the work done by the load is $\frac{1}{2}.w.\delta x.y$. The total work done by the loading is $\Sigma \frac{1}{2} w.\delta x.y$. Substituting for y and integrating over the whole range from $x = 0$ to $x = l$, we can work out the value of the work supplied. It is then easy to express it in terms of the maximum bending stress f and the dimensions of the beam.

It is however more convenient to derive an expression for the amount of work done in terms of the bending moment at a cross-section. If M is the bending moment on the element δx of the beam, the change of slope brought about is given by $EI \frac{d^2y}{dx^2} = EI \frac{di}{dx} = M$, i.e., the change of slope $\delta i = \frac{M \delta x}{EI}$. Now the work done by a couple is measured by the product of the couple and the angle turned through. In this case, since the loads are gradually applied, the work done on the element = average couple × angle turned through = $\frac{1}{2}M \times \delta i$. The total work supplied to the beam is, therefore,

$$\Sigma \tfrac{1}{2}M.\delta i = \tfrac{1}{2}\sum_{0}^{l} M.\frac{M\delta x}{EI} = \tfrac{1}{2}\int_{0}^{l}\frac{M^2dx}{EI}.$$

For a beam with a uniformly distributed load w,

$$M = -\frac{wl}{2}.x + \frac{wx^2}{2}.$$

The energy to be stored

$$= \frac{1}{2EI}\int_{0}^{l}\frac{w^2x^2(l-x)^2}{4}\,dx$$

$$= \frac{w^2}{8EI}\int_{0}^{l} x^2\,(l^2 - 2lx + x^2)\;dx = \frac{w^2}{8EI}\left\{l^2.\frac{l^3}{3} - 2l.\frac{l^4}{4} + \frac{l^5}{5}\right\}$$

$$= \frac{w^2l^5}{240EI}.\quad \text{Also } \frac{wl^2}{8} = fZ = \frac{2fI}{d} \quad \therefore \quad wl^2 = \frac{16fI}{d}.$$

Substituting for wl^2,

$$\text{Strain-energy} = \frac{l}{240EI} \times \frac{256f^2.I^2}{d^2} = \frac{16}{15} \times \frac{f^2}{E} \times \frac{Il}{d^2}$$

Putting $I = Ak^2$

$$= \frac{16}{15}\cdot\frac{k^2}{d^2}\cdot\frac{f^2}{E} \times \text{volume of the beam.}$$

For a rectangular section, $\frac{k^2}{d^2} = \frac{1}{12}$ and c, in this case, is $\frac{16}{15} \times \frac{1}{12}$ or $\frac{4}{45}$.

For a circular section, $\frac{k^2}{d^2} = \frac{1}{16}$ and $c = \frac{1}{15}$.

(iii) Beam of uniform section under a constant bending moment M.

$$\text{Strain-energy} = \int_{0}^{l}\frac{M^2dx}{2EI} = \frac{M^2.l}{2EI}$$

Also, $M = fZ = \dfrac{2fI}{d}$.

Substituting for M,

$$\text{Strain-energy} = \frac{l}{2EI} \times \frac{4f^2.I^2}{d^2} = \frac{2.f^2}{E}.\frac{I.l}{d^2}$$

$$= 2 \times \frac{k^2}{d^2} \times \frac{f^2}{E} \times \text{volume of the beam.}$$

For a rectangular section, $\dfrac{k^2}{d^2} = \frac{1}{12}$ and $c = \frac{1}{6}$.

For a circular section $\dfrac{k^2}{d^2} = \frac{1}{16}$ and $c = \frac{1}{8}$.

17. Deflection of beams by consideration of strain-energy: At a distance x from the origin consider a section X where the bending moment is M. At a distance $(x + \delta x)$ from the origin consider a contiguous section where the B.M. is $(M+\delta M)$. If δx is made very small, the bending moment over the strip will be sensibly M. Imagine an elementary point load δW at a section X_1 distant x_1 from the origin, where the deflection y_1 is to be evaluated. Then the work supplied by $\delta W = \frac{1}{2}\delta W.y_1$. This is stored. The change of slope brought about on the elementary strip δx due to this elementary load δW is given by $EI\dfrac{\delta i}{\delta x} = m.\delta W$, where m is the bending moment at the section X due to a point-load of unity at X_1.

$$\therefore \quad \delta i = \frac{m.\delta W.\delta x}{EI}.$$

The work stored by the strip δx is equal to the average couple × angle turned through. The bending moment at the section X is $(M + m\delta W)$, the second term being due to the elementary load δW at X_1.

The work supplied by δW and stored by the beam

$$= \Sigma\, \tfrac{1}{2}\, (M + m.\delta W)\ \delta i$$

$$= \Sigma\, \tfrac{1}{2}\, \frac{(M + m.\delta W)\ m.\delta W.\delta x}{EI}$$

$$= \frac{1}{2EI}\int_0^l M.m.\delta W.\delta x,$$ neglecting small quantities of a higher order.

Equating $\frac{1}{2}\,\delta W.y_1$ to $\frac{1}{2EI}\int_0^l M.m.\delta W.\delta x,$

we get, $y_1 = \int_0^l \frac{M.m.dx}{EI}.$

This expression enables us to evaluate the deflection at any section of the beam.

18. Strain-energy due to shear in a beam: Consider an element of length δx of a beam, at a distance x from the origin, where the shear force is F. The intensity of shear stress at a distance y from the N.A. in the section is given by, $q = \frac{F}{Iz}\sum_{y}^{y_c} y.\delta a.$

The amount of strain-energy stored by a strip of thickness δy will, then be

$$= \frac{q^2}{2N} \times \text{volume of the strip}$$

$= \frac{q^2}{2N} \times z.\delta y.\delta x.$ The total energy stored by the element δx of the beam is,

$\sum_{y_t}^{y_c} \frac{q^2}{2N} \times z\delta y.\delta x.$ The total amount of energy stored by the beam due to shear will be,

$$\int_0^l \int_{y_t}^{y_c} \frac{q^2}{2N} \times z\, dy.dx.$$

We shall apply the formula to work out the shear strain-energy of a simply supported beam of rectangular section $b \times d$, carrying a central load W on a span l. At any section,

$$F = \frac{W}{2}.$$

$$q = \frac{F}{I.b}\times b\left(\frac{d}{2}-y\right)\times\tfrac{1}{2}\left(\frac{d}{2}+y\right)=\frac{F}{2I}\left(\frac{d^2}{4}-y^2\right)=\frac{3W}{bd^3}\left(\frac{d^2}{4}-y^2\right)$$

$$\therefore \quad q^2 = \frac{9W^2}{b^2d^6}\left(\frac{d^4}{16}-\frac{d^2y^2}{2}+y^4\right)$$

Then, the strain-energy due to shear

$$= 2\int_0^{l/2}\int_{-d/2}^{+d/2}\frac{1}{2N}\times\frac{9W^2}{b^2d^6}\left(\frac{d^4}{16}-\frac{d^2y^2}{2}+y^4\right)b.dy.dx$$

$$= 2\int_0^{l/2}\frac{1}{2N}\times\frac{9W^2}{bd^6}dx\left\{\frac{d^4}{16}y-\frac{d^2y^3}{6}+\frac{y^5}{5}\right\}_{-d/2}^{+d/2}$$

$$= 2\int_0^{l/2}\frac{1}{2N}\times\frac{9W^2}{bd^6}dx\left\{\frac{d^5}{16}-\frac{d^2}{6}.\frac{d^3}{4}+\frac{1}{5}.\frac{d^5}{16}\right\}$$

$$= \frac{9W^2}{Nbd^6}\times\frac{d^5}{30}\int_0^{l/2}dx = \frac{9}{30}\times\frac{W^2}{Nbd}\times\frac{l}{2}=\frac{3}{20}\times\frac{W^2l}{N.bd}$$

If δ be the central deflection due to shear, the work supplied $= \frac{1}{2}W.\delta_s$.

$$\text{Equating,}\quad \tfrac{1}{2}\,W.\delta_s = \frac{3}{20}\times\frac{W^2.l}{N.bd}$$

$$\delta_s = \frac{3}{10}\times\frac{Wl}{N.b.d}$$

The total deflection at the centre due to the B.M. and S.F. will be

$$= \frac{Wl^3}{48EI}+\frac{3}{10}.\frac{Wl}{Nbd}=\frac{Wl^3}{Eb.d^3}\left\{\frac{1}{4}+\frac{3}{10}.\frac{E}{N}.\frac{d^2}{l^2}\right\}$$

19. Laminated Springs: These are built-up of a number of plates, as shown in fig. 134 and are widely used for railway waggons and coaches and road-vehicles. They are loaded at the ends and supported in the middle. Essentially, they are beams of uniform strength and are given such initial curvature that the central deflection disappears and the plates become flat when the full load for which the spring has been designed, comes upon it.

Let l be the horizontal span of the spring; let n be the number of plates of width b thickness t. If the load is W, the maximum bending moment is $\frac{Wl}{4}$ at the centre. This is to be resisted by n plates. If f is the skin stress of the plate section, the moment of resistance of each plate is $f \times \frac{1}{6}\,bt^2$. n plates will therefore offer a resistance of $n \times f \times \frac{1}{6}\,bt^2$.

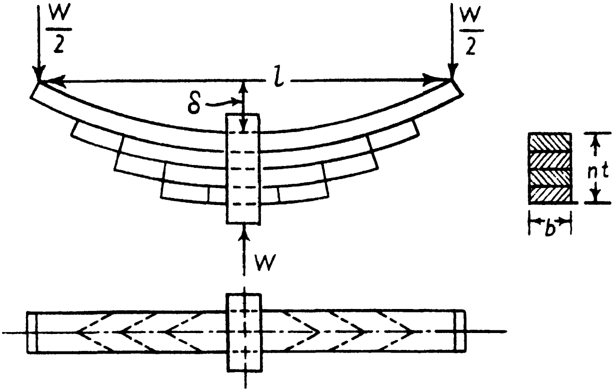


FIG. 134

Equating,

$$\frac{Wl}{4} = n \times f \times \tfrac{1}{6}bt^2,$$ from which the maximum stress induced can be worked out.

The plates are bent to a circular arc. The radius R to which the top plate should be bent or its central deflection which should be initially given so that the plates become flat when the full load W is put, can be easily calculated. We know that,

$$\delta = \frac{l^2}{8R}$$

Also, $$\frac{p}{y} = \frac{f}{t/2} = \frac{E}{R}$$

$$\therefore \quad R = \frac{Et}{2f} \text{ and } \delta = \frac{fl^2}{4E.t}$$

Example 11.

A carriage spring is to be 3 feet long and built-up of plates 3" wide and $\frac{3}{8}$" thick. Calculate the number of plates required for the spring if a central load of 1500 lb. is to be carried without exceeding the bending stress of 12 tons/in^2. What should be the central deflection and the initial radius to which the plates must be bent so that they straighten out under the load? $E = 30 \times 10^6$ lb/in^2.

$$\text{Maximum B.M.} = \frac{Wl}{4} \times 12 \text{ lb-inches.}$$

$$1500 \times 3 \times 12$$

$$4$$

$$= 13500 \text{ lb-inches.}$$

Moment of resistance of one plate

$$= f \times \tfrac{1}{6} bt^2$$

$$= 12 \times \tfrac{1}{6} \times 3 \times (\tfrac{3}{8})^2 \times 2240$$

$$= 1890 \text{ lb-inches.}$$

Equating, $13500 = n \times 1890$

$\therefore \quad n = 7{\cdot}1$, say, 8.

Providing 8 plates, the skin stress in each plate due to a load of 1500 lb. will be given by,

$$13500 = 8 \times f \times \tfrac{1}{6} \times 3 \times \tfrac{9}{64}$$

$\therefore \quad f = 24000$ lb/in^2. against 12 tons/in^2., permitted.

$$\frac{E}{R} = \frac{p}{y} = \frac{f}{t/2} = \frac{24000 \times 16}{3} = 128000$$

$$\therefore \quad R = \frac{30 \times 10^6}{128000} = 234{\cdot}375 \text{ inches}$$

$$\text{and } \delta = \frac{l^2}{8R} = \frac{36 \times 36 \times 128000}{8 \times 30 \times 10^6} = {\cdot}6912 \text{ inch.}$$

20. Beams of variable section: If the beam has a varying section, the relation $EI\frac{d^2y}{dx^2} = M$, may be re-arranged as $E\frac{d^2y}{dx^2} = \frac{M}{I}$; successive integrations will give the slope and deflection if $\frac{M}{I}$ is an integrable function of x.

Example 12.

A freely supported beam of span l carries a central load W. The sectional area of the beam is so designed that the moment of inertia of the section increases uniformly from I at the ends to 2I at the middle. Calculate the central deflection.

Take the origin at the left hand support. At a section X, at a distance of x from the origin, in the left hand half of the span, $M = -\frac{W}{2}.x$. The moment of Inertia of the section

$$I_x = I + I \times \frac{x}{l/2} = I\left(1 + \frac{2x}{l}\right)$$

Now, $EI_x\frac{d^2y}{dx^2} = -\frac{W}{2}x$

$$\therefore \quad E\frac{d^2y}{dx^2} = -\frac{Wx}{2I\left(1+\frac{2x}{l}\right)} = -\frac{W}{2I}\left\{\frac{\frac{l}{2}\left(1+\frac{2x}{l}\right)-\frac{l}{2}}{\left(1+\frac{2x}{l}\right)}\right\}$$

$$= -\frac{Wl}{4I} + \frac{Wl}{4I\left(1+\frac{2x}{l}\right)}.$$

Integrating,

$$E\frac{dy}{dx} = -\frac{Wlx}{4I} + \frac{Wl^2}{8I}\log_e\left(1+\frac{2x}{l}\right) + C_1$$

At the centre, $x = \frac{l}{2}$ and $\frac{dy}{dx} = 0$

$$\therefore \quad 0 = -\frac{Wl^2}{8I} + \frac{Wl^2}{8I}\log_e 2 + C_1.$$

Substituting for C_1,

$$E\frac{dy}{dx} = -\frac{Wlx}{4I} + \frac{Wl^2}{8I} + \frac{Wl^2}{8I}\log_e\left\{\frac{1+\frac{2x}{l}}{2}\right\}$$

Integrating once again,

$$E.y = -\frac{Wlx^2}{8I} + \frac{Wl^2x}{8I}$$

$$+\frac{Wl^2}{8I}\left\{x\log_e\left(\frac{1}{2}+\frac{x}{l}\right) - \int\frac{\frac{x}{l}}{\left(\frac{1}{2}+\frac{x}{l}\right)}dx\right\} + C_2$$

$$= -\frac{Wlx^2}{8I} + \frac{Wl^2x}{8I}$$
$$+ \frac{Wl^2}{8I}\left[x\log_e\left(\frac{1}{2}+\frac{x}{l}\right) - \int\left\{1 - \frac{1}{2\left(\frac{1}{2}+\frac{x}{l}\right)}\right\}dx\right] + C_2$$

$$= -\frac{Wlx^2}{8I} + \frac{Wl^2x}{8I}$$
$$+ \frac{Wl^2}{8I}\left\{x\log_e\left(\frac{1}{2}+\frac{x}{l}\right) - x + \frac{l}{2}\log_e\left(\frac{1}{2}+\frac{x}{l}\right)\right\} + C_2.$$

At the origin, $x = 0$ and $y = 0$.

$$\therefore \quad 0 = \frac{Wl^2}{8I} \times \frac{l}{2}\log_e\frac{1}{2} + C_2.$$

Substituting for C_2,

$$E.y = -\frac{Wlx^2}{8I} + \frac{Wl^2}{8I}\left\{x\log_e\left(\frac{1}{2}+\frac{x}{l}\right) + \frac{l}{2}\log_e\left(1+\frac{2x}{l}\right)\right\}$$

At the centre, $x = \frac{l}{2}$ and $E\delta = -\frac{Wl^3}{32I} + \frac{Wl^3}{16I}\log_e 2$

$$\therefore \quad E.\delta = \frac{Wl^3}{16I}\left\{\log_e 2 - \frac{1}{2}\right\} = \frac{Wl^3}{16I}\left\{\cdot 6931 - \cdot 5\right\}$$

$$\therefore \quad \delta = \frac{\cdot 1931\,Wl^3}{16EI} = \frac{\cdot 01207\,Wl^3}{EI}$$

21. Graphical methods: We have seen the relation between the rate of loading on a beam at a given section and the S.F., the B.M., slope and deflection at the section. It is one of successive integrations and can be utilised to obtain them graphically.

Consider a beam of span l freely supported at the end with loading as shown in fig. 135. At any cross-section X at a distance of x from A,

$$\frac{dF}{dx} = w.$$

Integrating between A and X,

$$F_x - F_A = \int_0^x w\,dx$$

$$\therefore \quad F_x = F_A + \int_0^x w dx$$

$\therefore \quad F_x = -R_A +$ area of the load diagram between A and X. The S.F. at any cross-section can then be worked out and plotted to a suitable scale.

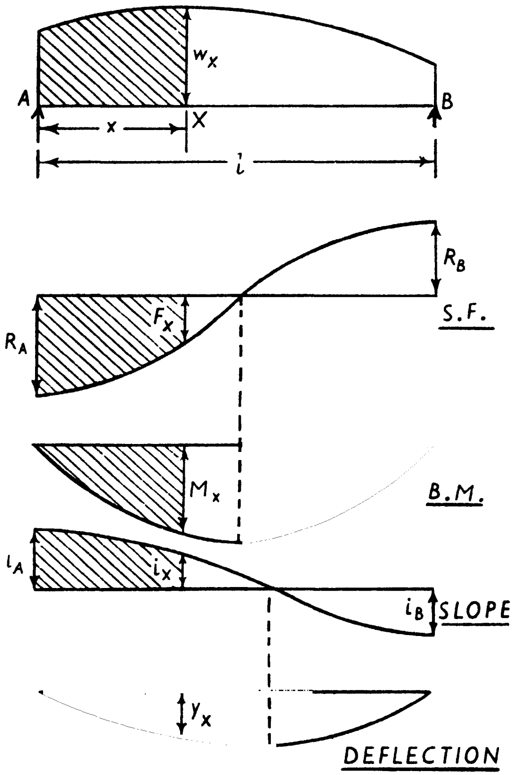


FIG. 135

Proceeding further,

$$\frac{dM}{dx} = F$$

$$\therefore \quad \delta M = F.\delta x.$$

Integrating between A and X,

$$M_x - M_A = \int_0^x F.dx.$$

$$\therefore \quad M_x = M_A + \int_0^x F.dx.$$

Since the beam is freely supported at the ends,

$$M_A = 0$$

$$\therefore \quad M_x = \int_0^x F.\delta x = \text{area of the S.F. diagram between } A \text{ and } X.$$

The B.M. at any section is now known and can be plotted.

Since $EI\dfrac{d^2y}{dx^2} = M$,

integrating between 0 to x,

$$EI\,(i_x - i_A) = \int_0^x Mdx$$

$$\therefore \quad i_x = i_A + \frac{1}{EI} \times \text{area of the B.M. diagram between } A \text{ and } X.$$

We have shown in article 14, how to obtain i_A, the slope at the support A. Therefore the slope i_x at any section can now be obtained and plotted.

Proceeding, $\dfrac{dy}{dx} = i$

$$\therefore \quad dy = i.dx.$$

Integrating between A and X,

$$y = \int_0^x i.dx = \text{area of the slope diagram between } A \text{ and } X.$$

The deflection at any section can thus be obtained from the slope diagram and plotted. All the diagrams are shown in fig. 135.

There is another graphical method which suggests itself. In Art. 6 of Chap. IV, we have seen that a funicular polygon for a given load system on a beam represents, to some scale, the B.M. diagram. The proposition was proved for point-loads but is is true for any manner of loading, because all loads, uniform or variable, can be looked upon as a series of point-loads.

We have seen that the relation between the rate of loading w and the bending moment M at a cross-section is $\frac{d^2M}{dx^2} = w$, for which, therefore, the graphical relation is that of a funicular polygon for the load system.

Now, $EI\frac{d^2y}{dx^2} = M$. It is clear that the relation between $EI.y$ and M is exactly the same as that between M and w. Therefore, if we treat the B.M. diagram as if it were a load system on a beam, and draw a funicular polygon for it, the vertical intercept of this funicular polygon at a section, will, to some scale, represent EIy and therefore the deflection of the beam at the section.

EXAMPLES VI

(1) Calculate the uniform bending moment which must be applied to a steel rod $\frac{1}{2}''$ in diameter in order to bend it into a circular arc of 50 feet radius. If the bar is 10 feet long, calculate the central deflection, taking $E = 13300$ tons/in². (152·4 lb-in; 3″.)

(2) A cantilever 6 feet long carries a load of 2 tons at the free end and 3 tons at the middle. By Macaulay's method or otherwise, calculate the slope and deflection at the free end, given $E = 13000$ tons/in². and $I = 375$ in⁴. (·001462 radian; ·075″.)

(3) A beam of cast iron, $1\frac{1}{2}''$ wide and 3″ deep is tested on supports 3 feet apart and shows a deflection of $\frac{1}{10}$ inch under a central load of 2 tons. Calculate E for the material. (5760 tons/in².)

(4) A beam of 20 feet span is freely supported and carries a point load at the centre. If the ends slope $\frac{1}{2}°$ under the load, calculate the maximum deflection. (·7″.)

(5) Three beams, each 12 feet long, placed side by side are freely supported at the ends and carry a central load of W tons. The two outside beams are 3″ wide and 6″ deep while the central one is 6″ wide and 3″ deep. If the stress in any beam is not to exceed $1\frac{1}{2}$ tons/in²., evaluate W and the load carried by each beam.

$$\left(1\,\frac{11}{16}\text{ tons; }\tfrac{3}{4}\text{ ton on each outer joist; }\frac{3}{16}\text{ ton on inner joist.}\right)$$

(6) A simply supported beam of rectangular section carries a central load of 5000 lb. over a span of 20 feet at a stress of 1000 lb/in². Calculate the maximum deflection, given $E=1{\cdot}75\times10^6$ lb/in². and the depth of section $=16''$. Calculate also the necessary width of section. (·343″; 7″.)

(7) A beam of length $2l$ carries a uniformly distributed load w per unit run and rests on three supports, two at the ends and one in the middle. Find how much must the middle support be lower than the end ones in order that the pressures on the three supports shall be equal.

$$\left(\frac{7}{72}\cdot\frac{wl^4}{EI}\cdot\right)$$

(8) A cantilever of span l carries a point load W at half its length from the fixed end. The free end is supported to the level of the fixed end. Using Macaulay's method, find the reaction at the prop and the position and amount of maximum deflection.

$$\left(\frac{5W}{16};\ \frac{l}{\sqrt{5}}\text{ from free end; }\frac{wl^3}{48\sqrt{5}EI}\cdot\right)$$

(9) A cantilever of span l is fixed at one end, the other resting freely on the middle of a simply supported cross-beam of the same span and section. If the cantilever is now loaded with a uniform load of w per unit run, find the reaction at its free end, offered by the cross-girder.

$$\left(\frac{6}{17}wl.\right)$$

(10) A beam of rectangular section is freely supported at the ends and carries a uniformly distributed load. If the maximum stress due to bending is to be $\frac{1}{2}$ ton/in²., determine the ratio of depth of section to span so that the maximum deflection may not exceed $\frac{1}{480}$ of the span. $E=750$ tons/in². (Not less than 1/15.)

(11) Three rolled steel joists, placed side by side are freely supported over a span of 20 feet. The two outer ones are $12''\times5''\times32$ lb., each having $I_{xx}=221$ in⁴., the inner one being $12''\times6''\times54$ lb. with $I_{xx}=376$ in⁴. A concentrated load of 15 tons is so arranged at the centre of the span that the deflection of each joist is the same. Calculate the load carried by, and the stress induced in each joist.

$$\left\{\begin{array}{l}6{\cdot}9\text{ tons on inner joist; }4{\cdot}05\text{ tons}\\ \text{on each outer joist; }6{\cdot}6\text{ tons/in}^2.\end{array}\right\}$$

(12) An 18″×7″×75 lb. R.S.B. is freely supported over a span of 25 feet. If the maximum deflection is not to exceed $\frac{1}{800}$ of the span, find the maximum permissible uniformly distributed load on it and the corresponding intensity of bending stress, given $I_{xx}=1149$ in⁴.; $E=13000$ tons/in². (·64 ton per ft. run; 4·68 tons/in².)

(13) A girder is of 50 feet span, 5 feet deep, and is freely supported at the ends. It carries a uniformly distributed load which produces a maximum stress of 7 tons/in². If $E=12{,}600$ tons/in²., find the maximum deflection. If the flanges had been so designed as to have a uniform flange stress of 7 tons/in². throughout, what would be the maximum deflection? (0·7″; ·833″.)

(14) A beam of span l is freely supported at the ends. From each end, at a distance of a, there is a concentrated load of W on the beam. Calculate the deflection under each load and at the centre.

$$\left\{\frac{Wa^2\,(3l-4a)}{6EI};\ \frac{Wa\,(3l^2-4a^2)}{24EI}\right\}$$

(15) A 16″×6″×62 lb. R.S.J. carries a load of 12 tons at quarter span. If it is simply supported over a span of 24 feet, calculate the position and amount of maximum deflection, given $E=13{,}500$ tons/in². and $I_{xx}=725{\cdot}7$ in⁴. (13·42′ from end; ·4268″.)

(16) The section of a beam freely supported over a span of 20 feet is a 15″×6″×59 lb. R.S.J. having a 10″×½″ plate riveted to each flange. Calculate the central deflection when the beam carries a uniformly distributed load of 2 tons per foot run. I_{xx} for the joist $=628{\cdot}9$ in⁴.; $E=13000$ tons/in². (·45″.)

(17) A beam simply supported over a span of 32 feet carries concentrated loads of 4, 2 and 6 tons at 5, 15 and 29 feet from the left hand support. Use Macaulay's method to determine the position and amount of maximum deflection. I for the section $=1670$ in⁴.; $E=13000$ tons/in².

(15·83 feet; ·297″.)

(18) A beam, 20 feet span, freely supported at the ends, carries a load of 8 tons uniformly spread over 5 feet from the centre towards the right hand support. Calculate the position and amount of maximum deflection. $I=375{\cdot}77$ in⁴.; $E=13000$ tons/in².

(10·61 feet from left support; ·422″.)

(19) A freely supported beam of span l carries a load which varies uniformly from zero at one end to w per unit run at the other end. For a section at distance of x from the zero-end, put down expressions for the rate of loading, S.F., B.M., slope and deflection.

Hence, or otherwise, construct the load, S.F., B.M., slope and deflection diagrams for the beam. State the position and amount of maximum deflection. ($x = \cdot 52l$; $\cdot 00653 \dfrac{wl^4}{EI}$.)

(20) A beam 1 inch square in section is freely supported over a span of 3 feet. A weight of 10 lb. is dropped on to the middle of the beam from a height of $1\frac{1}{2}$ inches. Calculate the maximum instantaneous stress and the maximum instantaneous deflection. $E = 30 \times 10^6$ lb/in^2. (15,552 lb/in^2.; ·112″.)

(21) A 12″×6″×54 lb. R.S. Joist is simply supported over a span of 15 feet. A weight of $\frac{1}{2}$ ton is dropped on to the middle of the beam, producing an instantaneous maximum stress of 7 tons/in^2. Calculate the height from which the weight was dropped and the maximum instantaneous deflection. $I = 375\cdot 77$ in^4.; $E = 13000$ tons in^2. (2·115″; ·242″.)

(22) A gradually applied load of 500 lb-wt., at the middle of a simply supported beam produces a deflection of ·02 inch. What will be the maximum instantaneous deflection produced by a weight of 100 lb-wt. dropped on to the middle of the beam from a height of 12 inches? (·314″.)

(23) A laminated carriage spring is 36″ long and made of plates 3″ wide and 3/8″ thick. How many plates will be required if the spring has to carry a central load of 1500 lb-wt. at a stress not exceeding 10 tons/in^2.? Estimate the central deflection. $E = 30 \times 10^6$ lb/in^2. (9 plates; ·6144″.)

(24) A laminated carriage spring 30″ long is made up of ten leaves of equal thickness and 2″ wide. Find the thickness of the leaves if the bending stress is limited to 16 tons/in^2. when the spring is loaded with 1000 lb. at the centre. To what radius should the leaves be initially bent for the spring to be flat under this load and what will be its central deflection? Take $E = 12800$ tons/in^2. ($\frac{1}{4}$″; 100″; 1·125″.)

(25) A cantilever of length 5 feet is of rectangular section and has a constant width of $4\frac{1}{2}$″ while the depth varies from 4 inches at the free end to 9″ at the fixed end. There is a point load of 1000 lb. at the free end. Calculate the deflection of the free end if $E = 1\cdot 5 \times 10^6$ lb/in^2. (·31″.)

(26) A carriage spring is built-up of 9 plates, each 3″ wide and $\frac{1}{4}$″ thick. Find the length of the spring so that it may carry a central load of 900 lb., the stress being limited to 10 tons/in^2. Neglecting friction, find also the deflection of the spring at the centre. $E = 30 \times 10^6$ lb/in^2. (L.U.)

(28″; ·5855″.)

(27) A 24″ × 7½″ rolled steel joist ($I_{xx}=2443$ in^4.) is simply supported at its ends on a span of 30 feet and carries a uniformly distributed load (including its own weight) of 0·4 ton per foot run. Given that,

(a) the maximum deflection must not exceed ½″, (b) the maximum stress due to bending must not exceed 8 tons/in^2.

Calculate the greatest value of an additional concentrated load which may be placed on the joist at the middle of the span. $E=12000$ tons/in^2. (L. U.)

(7·58 tons.)

(28) A horizontal cantilever of uniform section and length L carries a uniformly distributed load w per unit length throughout its length. The cantilever is supported by a rigid prop at a distance kL from the fixed end, the level of the beam at the prop being the same as that at the fixed end. Determine the value of k which will make the bending moment at the prop the same as that at the fixed end.

Sketch the S.F. and B.M. diagrams for the cantilever and show on them all maximum values. (L. U.)

$$\left\{k = \frac{\sqrt{6}}{1+\sqrt{6}} = \cdot 71.\right\}$$

(29) A cantilever of effective length l carries a total load wl uniformly distributed throughout the length. If the cantilever is propped at a point $\frac{l}{4}$ from the free end and the level at the prop adjusted so that there is no deflection at the free end, derive a formula for the reaction at the prop and also for the deflection of the beam at the prop. Sketch, approximately to scale, the S.F. and B.M. diagrams for the cantilever. (L. U.)

$$\left\{\frac{16}{27}wl;\ \frac{wl^4}{6144EI}\right\}$$

(30) A horizontal steel beam of I section rests on a rigid support at one end, the other end being supported by a vertical steel rod ¾″ diameter, whose upper end is rigidly held in a support 7′ 6″ above the end of the beam. The beam is an 8″ × 4″ B.S.B. for which the relevant I is 55·63 in^4., and the distance between its two points of supports is 9 feet. A load of 500 lb. falls on the beam at mid-span from a height of ¾″ above the beam. Determine the maximum stresses reached in the beam and rod and find also the deflection of the beam at mid-span measured from the unloaded position. $E=30\times 10^6$ lb/in^2. (L. U.)

$$\left\{\begin{array}{l}\text{In beam, 13100 lb/in}^2\text{.;}\\ \text{In rod, 7660 lb/in}^2\text{; }\cdot 1175''.\end{array}\right\}$$

(31) A cantilever of uniform section is 12 feet long. It is rigidly fixed at one end and is propped to the same level as the fixed end by a rigid prop 4 feet from the free end. The cantilever carries a load which varies uniformly from 4000 lb. per foot at the fixed end to zero at the free end. Calculate the load on the prop and the maximum positive and negative bending moments. (L. U.)

$$\left\{\begin{array}{l} 9467 \text{ lb-wt.; } M_{max}=+20267 \text{ at fixed end,} \\ -9700 \text{ lb-ft. at } 4{\cdot}463 \text{ feet from fixed end.} \end{array}\right\}$$

(32) A beam of uniform section and length l is simply supported at its ends and carries a distributed load which varies uniformly from zero value at each end to a maximum intensity of w per unit length at a section $\frac{l}{3}$ from the right-hand end.

Show that the maximum deflection occurs at a distance approximately $0{\cdot}01\ l$ from mid-span and find the deflection at mid-span in terms of w, l, E and I. (L. U.)

$$\left\{{\cdot}008108\frac{wl^4}{EI}\right\}$$

(33) A freely supported beam of span 24 feet carries a distributed load (including its own weight) which varies linearly from 0·4 ton/ft. at one end to 1 ton/ft. at the other. The beam is 18″ deep, its second moment of area is 842 in^4. and $E=13500$ tons/in^2. Find,

(a) the position and magnitude of maximum B.M.
(b) the maximum bending stress.
(c) the deflection at the centre of the span. (L. U.)

$$\left\{\begin{array}{l} \text{(a) } 12{\cdot}84 \text{ feet from left hand end, } 50{\cdot}59 \text{ ton-ft.} \\ \text{(b) } 6{\cdot}489 \text{ tons/in}^2\text{.; (c) } {\cdot}4596''. \end{array}\right\}$$

(34) A beam of rectangular section has a uniform breadth B and depth which varies uniformly from D at each end to $3D$ at the middle of its length. A second beam, made of the same material as the first beam, has the same length and breadth but a uniform depth $3D$. Find the ratio of the strain-energy of the first beam to that of the second beam when each is simply supported at its ends and carries a central point load W. (L. U.)

$$\left\{\frac{9}{8}(9\log_e 3-8)\right\}$$

(35) A timber cantilever of rectangular section is 3″ wide and 10″ deep at the fixed end and tapers uniformly to 3″ wide and 4″ deep at the free end. The projecting length is 6 feet and there is a load of 400 lb. at the extreme end.

Calculate the end deflection and the maximum bending stress. Take $E=2\times10^6$ lb/in^2. (L. U.)

(0·1872″; 600 lb/in^2. at 4 feet from free end.)

(36) An *I*-section consists of two flanges $12'' \times \frac{1}{2}''$ welded to a web $9'' \times 1''$. It is used as a cantilever 3 feet long to carry an end load. Estimate the relative magnitudes of the end deflections due to bending and shear if $\frac{E}{C} = \frac{5}{2}$ and find the actual values if $W = 2$ tons and $E = 30 \times 10^6$ lb/in². (L. U.)

($\delta_b : \delta_s :: 5{\cdot}081 : 1$; $\delta_b = {\cdot}007''$; $\delta_s = {\cdot}001378''$.)

(37) A thin flat plate of length L, breadth B and thickness t is built-in along its edges and subjected to a normal loading of w per unit area. From consideration of the central deflection, show that the maximum bending stress at the centre will be given approximately by

$$\frac{wB^2L^4}{4t^2(L^4+B^4)}$$ (L. U.)

(38) A steel girder ($I = 512$ in⁴.) is simply supported over a span of 24 feet. For the middle 12 feet of the span, the girder is reinforced with a plate on each flange, giving a total $I = 885$ in⁴. for this portion. The girder carries a concentrated load of 10 tons at mid-span together with 6 tons at a point 4 feet from mid-span. Determine the maximum deflection and where it occurs, neglecting the weight of the girder itself. $E = 13400$ tons/in². (L. U.)

(Between the loads at $0{\cdot}38'$ from centre; $0{\cdot}6961''$.)

CHAPTER VII

BEAMS AND BENDING — IV.

Fixed and Continuous Beams

1. Fixed beams: Fixed, built-in, or encastré beams are constrained at the supports so that the slopes at the supports when the beam is loaded, remain zero. This is possible only if restraining couples are exerted at the supports, their direction being opposite to the bending moments producing concave bending under a given system of loading. There is a unique value for the constraining couple at each support so that the support-slope is zero. If the actual moment at a support is less than this value, there will be some slope at the support; if more, the slope will be on the other side of the zero-position. It is necessary to work out the magnitudes of these fixing couples at the supports so that the slope at the supports is zero. It is then easy to put down the value of the bending moment at any section of a fixed beam and work out the slope and deflection at the section. This work is even easier than in the case of a freely supported beam.

2. Bending moment diagrams: It is best to look upon a fixed beam as equivalent to a freely supported beam to which are added fixing couples at the supports of such magnitudes that the slope is zero.

Consider a fixed beam with a given system of loads on it. The bending moment at any section X distant x from A will be given by, $M_x = \mu_x + \mu'_x$, where μ_x is the bending moment at the section due to the loading as for a freely supported beam and μ'_x is the bending moment due to the fixing couples M_A and M_B at the supports.

Referring to fig. 136(*iii*), $\mu'_x = M_A + (M_B - M_A)\frac{x}{l}$. ✓

As for μ_x, there is no difficulty about calculating it for a given system of loads. The bending moment M_x at any section of a fixed beam is therefore given by, $M_x = \mu_x + \mu'_x = \mu_x + M_A + (M_B - M_A)\frac{x}{l}$ and can be calculated as soon as the values of M_A and M_B are known. Fig. 136(*iv*) shows the B.M. diagram for a fixed beam. Since μ_x and μ'_x are opposite in sign, the B.M. diagram is partly positive and partly negative. On the portions on which the B.M. is positive, the beam bends convex upwards and where the B.M. is negative, it bends concave upwards.

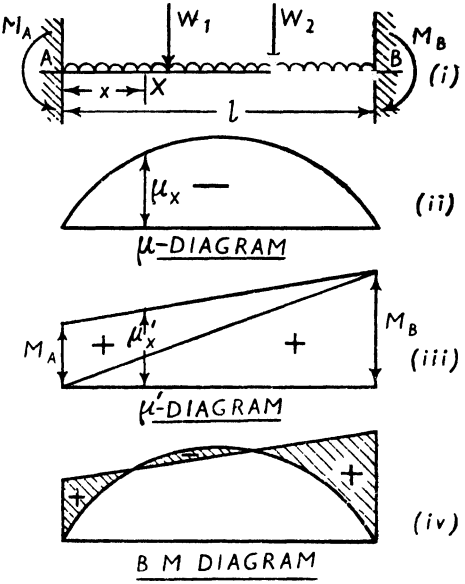


FIG. 136

3. Support-moments: To evaluate the support-moments for horizontal beams of uniform section, consider the bending moment at any section of the beam fig. 136(*i*).

Then, $M_x = EI\frac{d^2y}{dx^2} = \mu_x + \mu'_x$.

Integrating over the whole length from 0 to l,

$$EI\left[\frac{dy}{dx}\right]_0^l = \int_0^l \mu_x dx + \int_0^l \mu'_x dx.$$

$\therefore \quad EI\,(i_B - i_A) = a + a'$, where a is the area of the μ-diagram and a' is the area of the μ'-diagram.

Since the slopes at the supports are zero,
$i_A = i_B = 0$ and $a + a' = 0$.

The area under the μ-diagram is numerically equal to the area under the μ'-diagram, since their algebraic sum $a + a'$ is zero. The area of the μ'-diagram which is a trapezium is $(M_A + M_B).\frac{l}{2}$.

Equating numerically,

$$(M_A + M_B)\frac{l}{2} = -\,a \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

This provides one relation to evaluate the couples M_A and M_B. For one more equation that is necessary, multiply both sides of the general equation of moments by x and then integrate over the whole range. We have,

$$EI\int_0^l x\frac{d^2y}{dx^2}\,dx = \int_0^l \mu_x.x.dx + \int_0^l \mu'_x.x.dx.$$

$\therefore \quad EI\left\{x\frac{dy}{dx} - y\right\}_0^l = a\bar{x} + a'\bar{x}'$, where $\bar{x}$ and $\bar{x}'$ are the distances of the centres of gravity of the diagrams from the origin.

$$\therefore \quad EI\left\{(li_B \quad y_B) - (0.i_A - y_A)\right\} = a\bar{x} + a'\bar{x}'.$$

Since, i_A, i_B, y_A, y_B are all zero,
$a\bar{x} + a'\bar{x}' = 0$.

Therefore, the moments of the μ and μ'-diagrams about the origin are numerically equal. Since the areas are also numerically equal, it is clear that $\bar{x} = \bar{x}'$ i.e. their centres of gravity lie on the same vertical line.

The moment of the μ'-diagram about the origin A is, $M_A \times \frac{l}{2} \times \frac{l}{3} + M_B \times \frac{l}{2} \times \frac{2l}{3} = (M_A + 2M_B)\,\frac{l^2}{6}$, if we split up the trapezium into two triangles.

$$\therefore \quad (M_A + 2M_B)\frac{l^2}{6} = -a\bar{x}$$

or $$(M_A + 2M_B) = -\frac{6a\bar{x}}{l^2} \qquad \text{(2)}$$

With the help of equations (1) and (2), we solve for M_A and M_B. When these have been evaluated, the bending moment at any section of the beam is known and successive integrations will give the slope and deflection at the section. We shall now take a few standard cases.

(i) Fixed beam, span l, central load W.

By symmetry, the two end-couples are equal and the μ'–diagram is a rectangle. The μ–diagram is a triangle with the central ordinate of $\frac{Wl}{4}$.

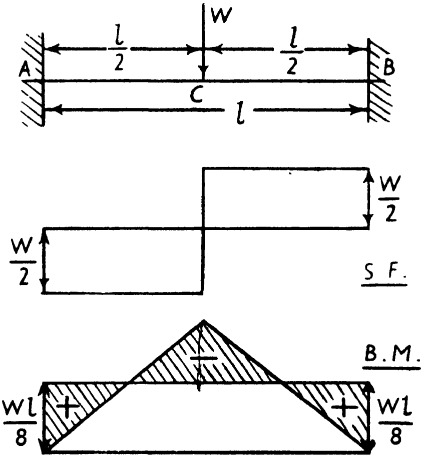


FIG. 137

Equating the areas a and a',

$$M_A \times l = \frac{1}{2} l \times \frac{Wl}{4}$$

$$\therefore \quad M_A = \frac{Wl}{8} = M_B.$$

The S.F. and B.M. diagrams are shown in fig. 137. The S.F. diagram is the same as that for a freely supported beam. The end-couples being equal and opposite balance each other and impose no additional reactions at the supports.

The points of contraflexure occur at $\frac{l}{4}$ from the ends, as is clear from the B.M. diagram.

For slope and deflection at any section in the portion *AC*, distant *x* from *A*,

$$EI\frac{d^2y}{dx^2} = \mu_x + \mu'_x = -\frac{W}{2}x + M_A = -\frac{W}{2}x + \frac{Wl}{8}.$$

Integrating,

$$EI\frac{dy}{dx} = -\frac{Wx^2}{4} + \frac{Wlx}{8} + 0, \text{ since, } \frac{dy}{dx} = 0 \text{ at } x = 0.$$

Integrating once more,

$$EIy = -\frac{Wx^3}{12} + \frac{Wlx^2}{16} + 0, \text{ since } y = 0 \text{ at } x = 0.$$

We have thus got the equations for slope and deflection at any section in *AC*. The maximum deflection occurs at the centre and,

$$EIy_C = -\frac{Wl^3}{96} + \frac{Wl^3}{64} = \frac{Wl^3}{192}$$

$$\therefore\ y_C = \frac{Wl^3}{192EI} \text{ or } \tfrac{1}{4} \text{ that for a freely supported beam.}$$

The effect of the restraining couples at the supports is to reduce the bending moment throughout, the maximum being half the value for a freely supported beam. Consequently, the slopes and deflections are reduced throughout. The obvious effect of fixation on a beam of a given section is therefore, to make it stronger as well as stiffer.

(ii) Span *l*, uniformly distributed load *w*.

The fixing couples are equal by symmetry. The μ-diagram is a parabola with a central ordinate of $\frac{wl^2}{8}$. Equating the two areas *a* and *a'*,

$$M_A.l = \tfrac{2}{3}.\frac{wl^2}{8}.l$$

$$\therefore\quad M_A = \frac{wl^2}{12} = M_B.$$

The bending moment at any section, distant x from the left-hand support is,

$$M_x = \mu_x + \mu'_x = -\frac{wl}{2}x + \frac{wx^2}{2} + \frac{wl^2}{12}.$$

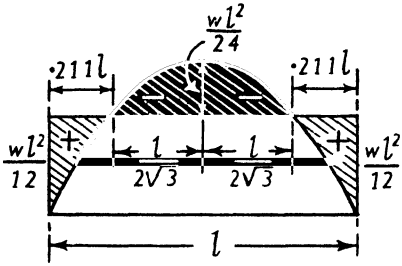


FIG. 138

For points of contraflexure,

$$\frac{wx^2}{2} - \frac{wlx}{2} + \frac{wl^2}{12} = 0$$

i.e. $$x^2 - lx + \frac{l^2}{6} = 0$$

$$\therefore \quad \left(x - \frac{l}{2}\right)^2 = \frac{l^2}{12}$$

$$\therefore \quad x = \frac{l}{2} \pm \frac{l}{2\sqrt{3}}$$

The points of contraflexure are at $\frac{l}{2\sqrt{3}}$ on either side of the centre. Their distances from the left hand support are

$$x = \cdot 5l \pm \cdot 289l$$
$$= \cdot 211l \text{ or } \cdot 789l.$$

For slope and deflection at any section,

$$EI\frac{d^2y}{dx^2} = -\frac{wlx}{2} + \frac{wx^2}{2} + \frac{wl^2}{12}.$$

Integrating,

$$EI\frac{dy}{dx} = -\frac{wlx^2}{4} + \frac{wx^3}{6} + \frac{wl^2}{12}.x + 0, \text{ since } \frac{dy}{dx} = 0 \text{ at } x = 0.$$

Integrating once again,

$$EIy = -\frac{wlx^3}{12} + \frac{wx^4}{24} + \frac{wl^2x^2}{24} + 0, \text{ since } y = 0 \text{ at } x = 0.$$

It will be noticed that the constants of integration give no trouble, $\frac{dy}{dx}$ and y both being zero at $x = 0$.

The maximum deflection is at the centre where,

$$EIy_C = -\frac{wl^4}{96} + \frac{wl^4}{384} + \frac{wl^4}{96} = \frac{wl^4}{384}$$

$$\therefore \quad y_C = \frac{1}{384} \times \frac{wl^4}{EI}$$ i.e. it is $\frac{1}{5}$ of that for a freely supported beam.

Example 1.

A fixed beam of 30 feet span carries two point loads of 18 tons at 10 feet from each end. Obtain the values of the support-moments and the central moment. Calculate also the deflection at the centre, given I for the section = 2654 in⁴. and E = 13000 tons/in².

The area of the μ-diagram $a = 2 \times \frac{10}{2} \times 180 + 180 \times 10$
$= 3600$ units.

The area of the μ'-diagram $a' = M_A \times 30$.

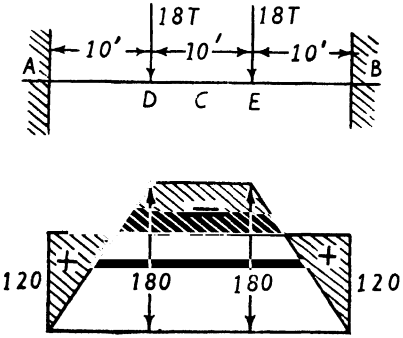


FIG. 139

Equating,

$M_A = +\ 120$ ton-feet.

The central moment $= -\ 180 + 120 = -\ 60$ ton-feet.

For points of contraflexure,

$$M_x = +120 - 18x = 0$$

or, $x = \frac{20}{3}$ feet from either end.

The law of bending moment for all sections from A to B (fig. 139) is,

$$EI\frac{d^2y}{dx^2} = +120 - 18x \quad + 18(x - 10) \quad + 18(x - 20).$$

Integrating,

$$EI\frac{dy}{dx} = +120x - 9x^2 + 0 \quad + 9(x - 10)^2 \quad + 9(x - 20)^2.$$

Integrating again,

$$EIy = +60x^2 - 3x^3 + 0 \quad + 3(x - 10)^3 \quad + 3(x - 20)^3,$$

the constants of integration being zero.

The deflection at the centre will be obtained by substituting $x = 15$, in the expression $EIy = +60x^2 - 3x^3 + 3(x - 10)^3$

$$\therefore \quad EIy_C = 60 \times 15^2 - 3 \times 15^3 + 3 \times 5^3$$
$$= 3750$$

$$\therefore \quad y_C = \frac{3750 \times 1728}{13000 \times 2654} = \cdot 1878''.$$

(iii) Span l, eccentric load W.

In this case, M_A and M_B will not be equal. To evaluate them, it will be necessary to equate the areas and also the moments of the μ and μ'-diagrams.

$$\mu_C = \frac{Wab}{l}.$$

Equating a to a',

$$(M_A + M_B)\frac{l}{2} = \frac{Wab}{l}\cdot\frac{l}{2}$$

$$\therefore \quad M_A + M_B = \frac{Wab}{l} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Equating the moments of a and a' about A, we get,

$$(M_A + 2M_B)\frac{l^2}{6} = \frac{Wab}{l} \times \frac{a}{2} \times \frac{2a}{3} + \frac{Wab}{l} \times \frac{b}{2} \times (a + \frac{b}{3})$$

$$= \frac{Wab}{6l}(2a^2 + 3ab + b^2)$$

$$= \frac{Wab\,(2a + b)}{6}, \text{ since } a + b = l$$

$$\therefore\; M_A + 2M_B = \frac{Wab\,(2a + b)}{l^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

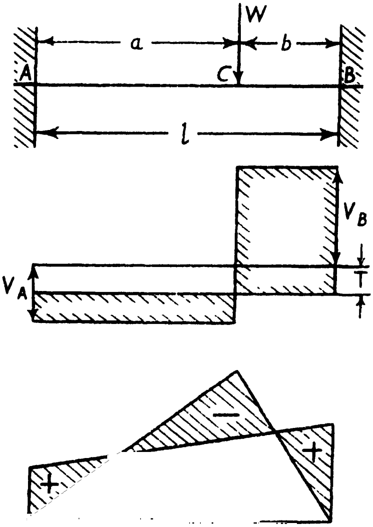


FIG. 140

Solving for M_A and M_B from (1) and (2),

$$M_B = \frac{Wab}{l}\left(\frac{2a + b}{l} - 1\right) = \frac{Wa^2b}{l^2} \text{ and } M_A = \frac{Wab^2}{l^2}.$$

For slope and deflection at any section in *AC* (fig. 140), we have,

$$EI\frac{d^2y}{dx^2} = \mu_x + \mu'_x = -\frac{Wbx}{l} + \left\{M_A + (M_B - M_A)\frac{x}{l}\right\}$$

$$= -\frac{Wbx}{l} + \frac{Wab^2}{l^2} + \frac{Wab\,(a - b)x}{l^3}.$$

Integrating,

$$EI\frac{dy}{dx} = -\frac{Wbx^2}{2l} + \frac{Wab^2}{l^2}.x + \frac{Wab\,(a - b)x^2}{2l^3} + 0,$$

since $\frac{dy}{dx} = 0$ at $x = 0$.

$$\therefore \quad EI\frac{dy}{dx} = \frac{Wab^2}{l^2}.x - \frac{Wb^2x^2(3a+b)}{2l^3}.$$

Integrating once again,

$$EIy = \frac{Wab^2x^2}{2l^2} - \frac{Wb^2(3a+b)x^3}{6l^3} + 0, \text{ since } y = 0 \text{ at } x = 0.$$

$$\therefore \quad EIy = \frac{Wb^2x^2}{6l^3}\{3al - x\ (3a + b)\}$$

For maximum deflection, $\frac{dy}{dx} = 0$

$$\therefore \quad \frac{Wab^2x}{l^2} - \frac{Wb^2x^2(3a+b)}{2l^3} = 0$$

or $$x = \frac{2al}{(3a+b)}.$$

Substituting in the expression for y,

$$EIy_{max} = \frac{Wb^2}{6l^3} \times \frac{4a^2l^2}{(3a+b)^2}\{3al - 2al\}$$

$$= \frac{2}{3}\frac{Wa^3b^2}{(3a+b)^2}.$$

The deflection under the load will be obtained by substituting $x = a$.

$$\therefore \quad EIy_C = \frac{Wb^2a^2}{6l^3}\{3a\ (a+b) - a\ (3a+b)\}$$

$$= \frac{Wa^3b^3}{3l^3}$$

$$\therefore \quad y_C = \frac{Wa^3b^3}{3EI.l^3}.$$

For the point of inflexion in AC,

$$M_x = -\frac{Wbx}{l} + \frac{Wab^2}{l^2} + \frac{Wab\ (a-b)x}{l^3} = 0$$

$$\therefore \quad x = \frac{al}{(3a+b)}.$$

Locate similarly the point of inflexion in CB.

The S.F. and B.M. diagrams are shown in fig. 140. For the S.F. diagram, the reactions at the supports must first be determined. If V_A and V_B are the reactions to the load W, as for a freely supported beam,

$V_A = \frac{Wb}{l}$ and $V_B = \frac{Wa}{l}$. To the unbalanced moment $(M_B - M_A)$, the reactions are a force T down at A and T upwards at B such that,

$$T.l = M_B - M_A = \frac{Wab\ (a-b)}{l^2}$$

$$\therefore \quad T = \frac{Wab\ (a-b)}{l^3}.$$

The reactions R_A and R_B at the supports are given by, $R_A = V_A - T$ and $R_B = V_B + T$.

The S.F. diagram is as for a freely supported beam with the base line shifted down by T.

Example 2.

A built-in beam of 20 feet span carries a concentrated load of 10 tons at 5 feet from the right hand support. Find the position and amount of maximum deflection, given I for the section $= 440{\cdot}5$ *in*^{4}*. and* $E = 13000$ *tons/in*^{2}*.*

$$\mu_C = \frac{Wab}{l} = \frac{10 \times 15 \times 5}{20} = 37{\cdot}5 \text{ ton-feet.}$$

Equating a and a',

$(M_A + M_B) = 37{\cdot}5$ ....(1)

Equating the moments of a and a',

$$(M_A + 2M_B).\ \tfrac{400}{6} = 37{\cdot}5 \times \tfrac{15}{2} \times \tfrac{2}{3} \times 15 + 37{\cdot}5 \times \tfrac{5}{2} \times (15 + \tfrac{1}{3} \times 5)$$

$$= \frac{37{\cdot}5 \times 700}{6}$$

$\therefore \quad M_A + 2M_B = 65{\cdot}625$ ....(2)

From (1) and (2),

$$M_B = 28{\cdot}125 \text{ ton-feet}$$

$$M_A = 9{\cdot}375 \text{ ton-feet.}$$

For slope and deflection at any section in the larger segment.

$$EI\frac{d^2y}{dx^2} = \mu_x + \mu'_x = -V_A.x + M_A + (M_B - M_A)\frac{x}{l}$$

$$= -\tfrac{5}{2}.x + 9{\cdot}375 + \frac{18{\cdot}75x}{20}$$

$$\therefore\quad EI\frac{d^2y}{dx^2} = 9{\cdot}375 - \frac{31{\cdot}25x}{20}$$

Integrating,

$$EI\frac{dy}{dx} = 9{\cdot}375x - \frac{31{\cdot}25x^2}{40}.$$

Integrating again,

$EIy = 9{\cdot}375\dfrac{x^2}{2} - \dfrac{31{\cdot}25x^3}{120}$, the constants of integration being zero.

For maximum deflection, $\dfrac{dy}{dx} = 0$

$$\therefore\quad 9{\cdot}375x - \frac{31{\cdot}25x^2}{40} = 0$$

or $\qquad x = 12$ feet.

$$EIy_{max} = \frac{144}{2}\left\{9{\cdot}375 - \frac{31{\cdot}25 \times 12}{60}\right\}$$

$$= 225$$

$$y_{max} = \frac{225 \times 1728}{440{\cdot}5 \times 13000} = {\cdot}06789''.$$

Example 3.

A built-in beam of 20 feet span carries a uniformly distributed load of $\frac{1}{2}$ ton per foot run over the left hand half of the span. Calculate the support-moments.

The reactions at the supports for a freely supported beam are $V_A = 3\frac{3}{4}$ tons and $V_B = 1\frac{1}{4}$ tons.

In the portion AC,

$$\mu_x = -\frac{15}{4}x + \frac{x^2}{4}$$

or $\left\{\frac{15}{4}x - \frac{x^2}{4}\right\}$ numerically since μ_x is otherwise negative.

$$\therefore \quad \mu_C = \frac{15}{4} \times 10 - \frac{100}{4} = 12{\cdot}5 \text{ ton-feet.}$$

The area of the μ-diagram, numerically, is,

$$a = \int_0^{10} \left(\frac{15x}{4} - \frac{x^2}{4}\right) dx + \tfrac{1}{2} \times 10 \times 12{\cdot}5$$

$$= \left\{\frac{15x^2}{8} - \frac{x^3}{12}\right\}_0^{10} + 62{\cdot}5.$$

$$= 104{\cdot}17 + 62{\cdot}5 = 166{\cdot}67 \text{ units.}$$

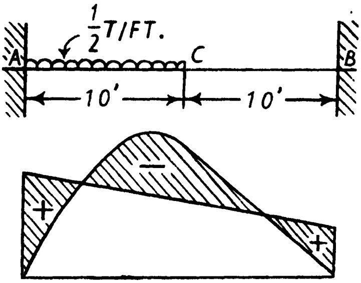


FIG. 141

The moment of the area of the μ-diagram about A, numerically, is,

$$a\bar{x} = \int_0^{10} \left(\frac{15x^2}{4} - \frac{x^3}{4}\right) dx + \tfrac{1}{2} \times 10 \times 12{\cdot}5 \times (10 + \tfrac{1}{3} \times 10)$$

$$= \left\{\frac{5x^3}{4} - \frac{x^4}{16}\right\}_0^{10} + 833{\cdot}33$$

$$= 1250 - 625 + 833{\cdot}33$$

$$= 1458{\cdot}33.$$

Equating a and a',

$$(M_A + M_B)\ 10 = 166{\cdot}67$$

$$\therefore \quad M_A + M_B = 16{\cdot}67.$$

Equating the moments of a and a',

$$(M_A + 2M_B)\,\frac{400}{6} = 1458{\cdot}33$$

$$\therefore \quad M_A + 2M_B = 21{\cdot}875.$$

Solving,

$$M_B = 5{\cdot}205 \text{ ton-feet}$$

$$M_A = 11{\cdot}465 \text{ ton-feet.}$$

The reactions at the supports to balance $(M_A - M_B) = 6{\cdot}26$ ton-feet, will be a force T upwards at A and downwards at B of magnitude $\frac{6{\cdot}26}{20}$ or ·313 ton. The reactions at the supports are, therefore,

$$R_A = V_A + T = 3{\cdot}75 + {\cdot}313 = 4{\cdot}063 \text{ tons upwards.}$$

$$R_B = V_B - T = 1{\cdot}25 - {\cdot}313 = 0{\cdot}937 \text{ tons upwards.}$$

4. Effect of sinking of a support: If one of the supports, say B, sinks by δ, we can calculate the effect at any section of the beam as under:

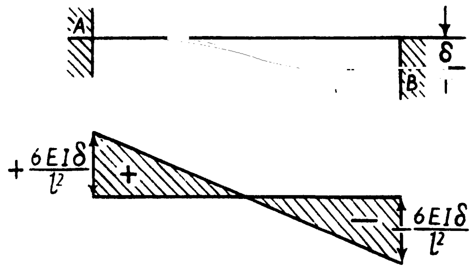


FIG. 142

The rate of loading at any section is zero. Taking the origin at A,

$$EI\,\frac{d^4y}{dx^4} = 0.$$

Integrating, $EI\,\frac{d^3y}{dx^3} = F_A$.

Integrating once again,

$EI\,\frac{d^2y}{dx^2} = F_A.x + M_A$, where F_A and M_A are respectively the S.F. and the B.M. at A.

Proceeding, $EI\dfrac{dy}{dx} = F_A\dfrac{x^2}{2} + M_A.x + 0,$

since $\dfrac{dy}{dx} = 0$ at A.

And $EIy = F_A.\dfrac{x^3}{6} + M_A\dfrac{x^2}{2} + 0,$ as $y = 0$ at $x = 0$.

At B, $x = l$ and $y = \delta$

$$\therefore \quad EI\delta = F_A.\frac{l^3}{6} + M_A.\frac{l^2}{2}$$

Also, at B, $\dfrac{dy}{dx} = 0$

$$\therefore \quad 0 = F_A.\frac{l^2}{2} + M_A.l.$$

Substituting for F_A, $M_A.l^2\left(\frac{1}{2} - \frac{1}{3}\right) = EI\delta$

$$\therefore \quad M_A = +\frac{6EI\delta}{l^2}$$

and $F_A = -\dfrac{2M_A}{l} = -\dfrac{12EI\delta}{l^3}$

$$M_B = F_A l + M_A = -\frac{12EI\delta}{l^2} + \frac{6EI\delta}{l^2} = -\frac{6EI\delta}{l^2}.$$

Thus, if the prop B sinks by δ below the level of A, there is a shear force throughout of $\dfrac{12EI\delta}{l^3}$, while the bending moment varies by a straight line law from $+\dfrac{6EI\delta}{l^2}$ at A to $-\dfrac{6EI\delta}{l^2}$ at B, as shown in fig. 142.

If a fixed beam which carries a uniformly distributed load has one support lower than the other by δ, the support-moments will be, $\dfrac{wl^2}{12} + \dfrac{6EI\delta}{l^2}$ at the higher and $\dfrac{wl^2}{12} - \dfrac{6EI\delta}{l^2}$ at the lower support.

5. Review of methods: In a fixed beam the relation between the areas of the μ and μ'-diagrams is so striking that it was used to obtain the magnitudes of the support-moments. Actually, the end conditions of the beam are adequate for evaluating the support-moments without any reference to the μ and μ'-diagrams.

Let us reconsider the case of a fixed beam carrying a central load *W*. The support-moments are equal by symmetry. Referring to fig. 137, the law of the bending moments from one end to the other, with *A* as origin, is given by,

$$EI\frac{d^2y}{dx^2} = M_A - \frac{W}{2}.x + W(x - \frac{l}{2}).$$

Integrating,

$$EI\frac{dy}{dx} = M_A.x \quad \frac{Wx^2}{4} + 0 + \frac{W}{2}(x - \frac{l}{2})^2$$

Integrating once, again,

$$EIy = M_A.\frac{x^2}{2} \quad \frac{Wx^3}{12} + 0 + \frac{W}{6}(x - \frac{l}{2})^3$$

The constants of integration are both zero since $\frac{dy}{dx}$ and *y* are zero at *A*.

At *B*, where $x = l, \frac{dy}{dx} = 0$

Substituting, we get,

$$0 = M_A.l - \frac{Wl^2}{4} + \frac{W}{2}(l - \frac{l}{2})^2$$

$$\therefore M_A = \frac{Wl}{8}, \text{ as before.}$$

Similarly, for a fixed beam carrying a uniformly distributed load *w* over the entire span,

$$EI\frac{d^2y}{dx^2} = M_A - \frac{wlx}{2} + \frac{wx^2}{2}$$

$$\therefore \quad EI\frac{dy}{dx} = M_A.x - \frac{wlx^2}{4} + \frac{wx^3}{6} + 0$$

and $$EIy = M_A.\frac{x^2}{2} - \frac{wlx^3}{12} + \frac{wx^4}{24} + 0.$$

At B, $\quad x = l$ and $\dfrac{dy}{dx} = 0$

$$\therefore \qquad 0 = M_A . l - \frac{wl^3}{4} + \frac{wl^3}{6}$$

$$\therefore \qquad M_A = \frac{wl^2}{12}.$$

For a fixed beam with an eccentric load W (fig. 140) we have, taking the origin at A,

$$EI\frac{d^2y}{dx^2} = M_A + (M_B - M_A)\frac{x}{l} - \frac{Wbx}{l} \quad + W(x-a)$$

$$\therefore \quad EI\frac{dy}{dx} = M_A . x + (M_B - M_A)\frac{x^2}{2l} - \frac{Wbx^2}{2l} + 0 \quad + \frac{W}{2}(x-a)^2$$

and $\quad EIy = M_A . \dfrac{x^2}{2} + (M_B - M_A)\dfrac{x^3}{6l} - \dfrac{Wbx^3}{6l} + 0 \quad + \dfrac{W}{6}(x-a)^3$

At B, where $x = l$, $\dfrac{dy}{dx} = 0$ and $y = 0$.

Substituting,

$$0 = M_A . l + (M_B - M_A)\frac{l}{2} - \frac{Wbl}{2} + \frac{W}{2}(l-a)^2$$

or, $\quad (M_A + M_B)\dfrac{l}{2} = \dfrac{Wbl}{2} - \dfrac{Wb^2}{2} = \dfrac{Wab}{2}$

$$\therefore \quad M_A + M_B = \frac{Wab}{l} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(1)}$$

Also, $0 = M_A . \dfrac{l^2}{2} + (M_B - M_A)\dfrac{l^2}{6} - \dfrac{Wbl^2}{6} + \dfrac{W}{6}(l-a)^3$

$$\therefore \quad \frac{l^2}{6}(2M_A + M_B) = \frac{Wbl^2}{6} - \frac{Wb^3}{6}$$

$$= \frac{Wb}{6}(l^2 - b^2) = \frac{Wab\,(a+2b)}{6}$$

$$\therefore \qquad 2M_A + M_B = \frac{Wab\,(a+2b)}{l^2} \qquad \ldots\ldots\ldots\ldots\ldots\ldots \text{(2)}$$

From (1) and (2),

$$M_A = \frac{Wab}{l}\left\{\frac{a+2b}{l} - 1\right\}$$

$$= \frac{Wab^2}{l^2}$$

Similarly,

$$M_B = \frac{Wa^2b}{l^2}, \text{ as before.}$$

6. Degree of restraint at supports for the maximum bending moment to be as small as possible: For a beam with a central load, the end restraints must be such that the end slopes are zero. The maximum positive moment will then be $\frac{Wl}{8}$ at the supports while the maximum negative moment will be at the centre and of the same magnitude. For a beam carrying a uniformly distributed load w, the end-moments must be $\frac{1}{2} \times \frac{wl^2}{8}$ or $+\frac{wl^2}{16}$ so that the central moment is of the same magnitude but opposite in sign. The end-slopes, in this case, will not be zero. With the origin at the left support,

$$EI\frac{d^2y}{dx^2} = +\frac{wl^2}{16} - \frac{wlx}{2} + \frac{wx^2}{2}$$

Integrating,

$$EI\frac{dy}{dx} = +\frac{wl^2}{16}.x - \frac{wlx^2}{4} + \frac{wx^3}{6} + C_1$$

At the centre, where $x = \frac{l}{2}$, $\frac{dy}{dx} = 0$

$$\therefore \quad 0 = +\frac{wl^3}{32} - \frac{wl^3}{16} + \frac{wl^3}{48} + C_1$$

or $$C_1 = \frac{wl^3}{96}$$

$$\therefore \quad EI\frac{dy}{dx} = +\frac{wl^2}{16}x - \frac{wlx^2}{4} + \frac{wx^3}{6} + \frac{wl^3}{96}$$

At A, where $x = 0$,

$$EIi_A = \frac{wl^3}{96}$$

$\therefore \quad i_A = \dfrac{wl^3}{96EI}$ i.e. the slope at the supports is one-fourth that for a freely supported beam.

Integrating the expression for slope,

$$EIy = +\frac{wl^2x^2}{32} - \frac{wlx^3}{12} + \frac{wx^4}{24} + \frac{wl^3}{96}x + 0.$$

At the centre,

$$EIy_C = +\frac{wl^4}{128} - \frac{wl^4}{96} + \frac{wl^4}{384} + \frac{wl^4}{192}$$

$$= \frac{wl^4}{192}$$

$$\therefore \quad y_C = \frac{wl^4}{192EI}.$$

The points of contraflexure will be obtained from

$$+\frac{wl^2}{16} - \frac{wlx}{2} + \frac{wx^2}{2} = 0$$

$$\text{or} \quad x^2 - lx + \frac{l^2}{8} = 0$$

$$\therefore \quad \left(x - \frac{l}{2}\right)^2 = \frac{l^2}{8}$$

$$\therefore \quad x = \frac{l}{2} \pm \frac{l}{2\sqrt{2}} = \frac{l}{2}(1 \pm \cdot 707) = \cdot 8535l \text{ or } \cdot 1465l.$$

7. Continuous beams: If a beam is continuous over intermediate supports, its curvature at the supports will be convex upwards and therefore the support-moments will be opposite in sign to those due to the load-system. The treatment of continuous beams will be similar to that of fixed beams and the bending moment diagram for a continuous beam under any loading will be obtained by treating it as a series of discontinuous beams from support to support, for which the usual μ–diagrams, as for freely supported beams, can be easily constructed and superimposing upon them the μ'–diagrams due to the support-moments, which will be of the opposite sign. As soon as the support-moments are evaluated, the B.M. diagram can be completed. The bending moment at any section will then be $\mu_x + \mu'_x$ and the slope and deflection can be obtained by successive integrations, as usual.

It must be noted, however, that the equations which were used in the case of fixed beams to evaluate the fixing couples at the supports are not applicable to continuous beams because the slopes at the supports are not necessarily zero.

To obtain the support-moments, we shall make use of Clapeyron's Theorem of Three Moments which connects up the support-moments at three consecutive supports of a continuous beam.

8. Clapeyron's Theorem of Three Moments:

I. Uniformly distributed loads.

Consider two consecutive spans AB, BC of a continuous beam. Let $AB = l_1$ and $BC = l_2$ and let w_1 and w_2 be the uniformly distributed loads per foot run on these spans respectively. Let M_A, M_B and M_C be the support-moments at A, B and C respectively, the supports being at the same level. Now, consider the span BC, taking the origin at B and x positive to the right. At a section, distant x from the origin,

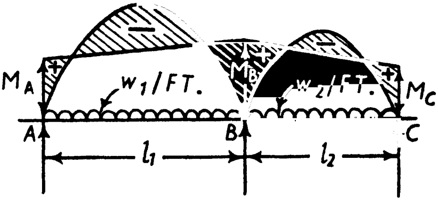


FIG. 143

$$EI\frac{d^2y}{dx^2} = \mu_x + \mu'_x = -\frac{w_2 l_2 x}{2} + \frac{w_2 x^2}{2} + M_B + (M_C - M_B)\frac{x}{l_2}.$$

Integrating,

$$EI\frac{dy}{dx} = -\frac{w_2 l_2 x^2}{4} + \frac{w_2 x^3}{6} + M_B.x + (M_C - M_B)\frac{x^2}{2l_2} + C_1.$$

At B, where $x = 0$, $\frac{dy}{dx} = i_B$

$$\therefore \qquad EIi_B = C_1.$$

The expression for slope is, therefore,

$$EI\frac{dy}{dx} = -\frac{w_2 l_2 x^2}{4} + \frac{w_2 x^3}{6} + M_B.x + (M_C - M_B)\frac{x^2}{2l_2} + EIi_B.$$

Integrating,

$$EIy = -\frac{w_2 l_2 x^3}{12} + \frac{w_2 x^4}{24} + M_B.\frac{x^2}{2} + (M_C - M_B)\frac{x^3}{6l_2} + EIi_B x + 0.$$

since $y = 0$ at $x = 0$.

At C, $x = l_2$ and $y = 0$

$$\therefore \quad 0 = -\frac{w_2 l_2^4}{12} + \frac{w_2 l_2^4}{24} + M_B.\frac{l_2^2}{2} + (M_C - M_B)\frac{l_2^2}{6} + EIi_B.l_2$$

$$\therefore \quad (2M_B + M_C)\frac{l_2}{6} - \frac{w_2 l_2^3}{24} + EIi_B = 0$$

$$\text{or } (2M_B + M_C)\, l_2 - \frac{w_2 l_2^3}{4} + 6EIi_B = 0 \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Next consider the span BA, taking the origin at B and x positive to the left.

Following the steps as above, we shall get,

$$(2M_B + M_A)\, l_1 - \frac{w_1 l_1^3}{4} + 6EIi'_B = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

But $i'_B = -i_B$ since the direction of x has been reversed. Adding (1) and (2) we get,

$$M_A.l_1 + 2M_B(l_1 + l_2) + M_C.l_2 = \frac{w_1 l_1^3}{4} + \frac{w_2 l_2^3}{4}.$$

This is the well-known theorem of three moments establishing a relation between the support-moments at three consecutive supports for a continuous beam carrying uniformly distributed loads.

If a beam has n supports, the end ones being free, there will be $(n - 2)$ intermediate supports and therefore $(n - 2)$ support-moments to determine, the moments at the free ends being zero. $(n - 2)$ equations are needed and they will be available on applying the theorem of three moments to consecutive pairs of spans—as *AB-BC*, *BC-CD*, *CD-DE* and so on.

We shall illustrate the method by applying it to a few simple cases.

(i) Two equal spans of l each: uniformly distributed load w on each.

We have already studied this case as that of a simply supported beam with a uniformly distributed load and propped at the middle to the level of the end supports. We shall now obtain the same results by using the theorem of three moments.

There is only one moment to be determined, viz., M_B, since the end moments M_A and M_C are zero. By the theorem of three moments,

$$M_A.l + 2M_B.2l + M_C.l = \frac{wl^3}{4} + \frac{wl^3}{4}$$

$$\therefore \quad 4M_B = \frac{wl^2}{2} \text{ or } M_B = \frac{wl^2}{8}.$$

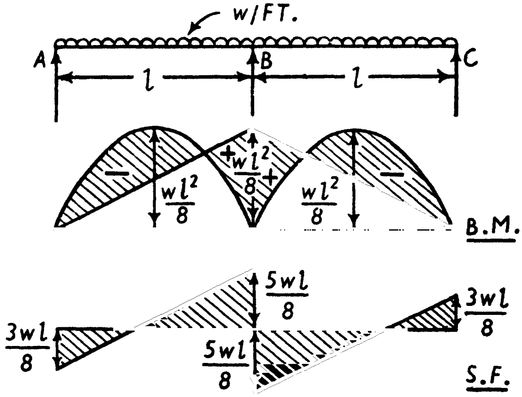


FIG. 144

To obtain the reactions at the supports, the bending moment at B is,

$$M_B = -R_A.l + wl.\frac{l}{2} = -R_A.l + \frac{wl^2}{2}.$$

But this is $+\dfrac{wl^2}{8}$.

Equating, $-R_A.l+\dfrac{wl^2}{2}=+\dfrac{wl^2}{8}$

$$\therefore \quad R_A=\frac{3wl}{8}=R_C \text{ by symmetry.}$$

The reaction R_B at the central prop is therefore, the balance i.e. $2wl-2\times\dfrac{3wl}{8}=\dfrac{5wl}{4}$.

The B.M. and S.F. diagrams are shown in fig. 144. To obtain the slope and deflection at any section, take the origin at *A*. At any section in *AB* at a distance of x from *A*,

$$EI\frac{d^2y}{dx^2}=-R_A.x+\frac{wx^2}{2}=-\tfrac{3}{8}wlx+\frac{wx^2}{2}$$

The point of inflection occurs where,

$$-\tfrac{3}{8}wlx+\frac{wx^2}{2}=0,\ \text{i.e. where } x=\frac{3l}{4}.$$

Integrating the expression for bending moment,

$$EI\frac{dy}{dx}=-\tfrac{3}{16}wlx^2+\frac{wx^3}{6}+C_1.$$

At B, $x=l$ and $\dfrac{dy}{dx}=0$, by symmetry.

$$\therefore \quad 0=-\frac{3wl^3}{16}+\frac{wl^3}{6}+C_1$$

$$\therefore \quad C_1=\frac{wl^3}{48}$$

Integrating once again,

$$EIy=-\tfrac{1}{16}wlx^3+\frac{wx^4}{24}+\frac{wl^3}{48}.x+0.$$

The slope and deflection anywhere in *AB* are therefore known. For maximum deflection, $\dfrac{dy}{dx}=0$

$$\therefore \quad -\tfrac{3}{16}wlx^2+\frac{wx^3}{6}+\frac{wl^3}{48}=0$$

$$\therefore \quad 8x^3 - 9lx^2 + l^3 = 0$$

$$\therefore \quad (l - x)\ (l^2 + lx - 8x^2) = 0$$

$$\therefore \quad 8x^2 - lx = l^2$$

$$\text{or} \quad x^2 - \frac{xl}{8} = \frac{l^2}{8}$$

$$\therefore \quad \left(x - \frac{l}{16}\right)^2 = \frac{l^2}{8} + \frac{l^2}{256} = \frac{33l^2}{256}$$

$$\therefore \quad x - \frac{l}{16} = \pm \frac{5{\cdot}745l}{16}$$

$$\therefore \quad x = \frac{6{\cdot}745}{16}.l = {\cdot}4216l$$

$$\therefore \quad x^3 \quad {\cdot}07494l^3 \text{ and } x^4 = {\cdot}03159l^4$$

$$\therefore \quad EIy_{max} = wl^4\left\{-\frac{{\cdot}07494}{16} + \frac{{\cdot}03159}{24} + \frac{{\cdot}4216}{48}\right\}$$

$$= wl^4[-{\cdot}00468 + {\cdot}00132 + {\cdot}0088]$$

$$= {\cdot}00544\ wl^1$$

$$\therefore \quad y_{max} = .00544\ \frac{wl^4}{EI}.$$

(ii) Three equal spans, each l; uniformly distributed load w on each.

$M_A = M_D = 0$, as the ends are freely supported.

$M_B = M_C$ by symmetry. We have therefore only one support-moment to evaluate. Applying the theorem of three moments to spans *AB-BC*, we get,

$$M_A \times l + 2M_B \times 2l + M_C \times l = \frac{wl^3}{4} + \frac{wl^3}{4}$$

$$\therefore \quad 5M_B = \frac{wl^2}{2} \text{ or } M_B = \frac{wl^2}{10} = M_C.$$

For reactions at the supports,

$$M_B = -R_A.l + \frac{wl^2}{2} = \frac{wl^2}{10}$$

$$\therefore \quad R_A = \frac{4wl}{10} = R_D \text{ by symmetry}$$

$$\therefore \quad R_B = R_C = \frac{11wl}{10}.$$

The B.M. and S.F. diagrams are shown in fig. 145. The slope and deflection at any section may now be obtained easily.

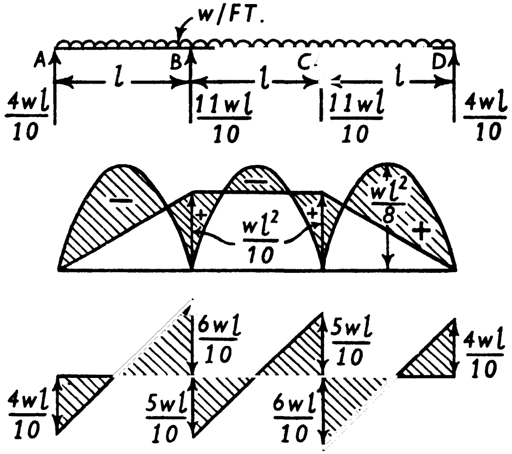


FIG. 145

(iii) Four equal spans, each l; uniformly distributed load w on each.

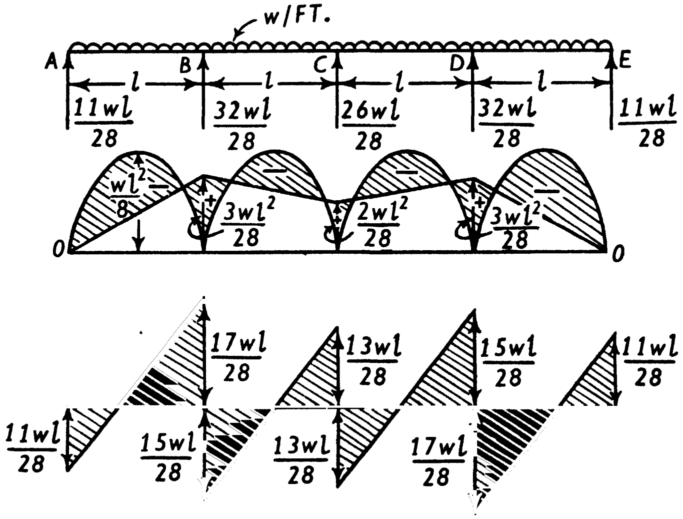


FIG. 146

In this case $M_B = M_D$ by symmetry and $M_A = M_E = 0$. We have therefore to determine M_B and M_C.

Taking the spans *AB-BC*,

$$M_A \times l + 2M_B \times 2l + M_C \times l = \frac{wl^3}{4} + \frac{wl^3}{4}$$

$$\therefore \quad 4M_B + M_C = \frac{wl^2}{2} \quad \ldots\ldots\ldots\ldots \quad (1)$$

Considering the spans *BC-CD*,

$$M_B \times l + 2M_C \times 2l + M_D \times l = \frac{wl^3}{4} + \frac{wl^3}{4}$$

$$\therefore \quad 2M_B + 4M_C = \frac{wl^2}{2} \quad \ldots\ldots\ldots\ldots \quad (2)$$

Solving for M_B and M_C from (1) and (2),

$$M_B = \frac{3}{28}wl^2 = M_D \text{ and } M_C = \frac{2wl^2}{28}.$$

For reactions at the supports,

$$M_B = -R_A.l + \frac{wl^2}{2} = \frac{3wl^2}{28}$$

$$R_A = \frac{11wl}{28} = R_E.$$

Also,

$$M_C = -R_A \times 2l - R_B \times l + 2wl.l = \frac{2wl^2}{28}$$

$$\therefore \quad R_B = -\frac{22wl}{28} + 2wl - \frac{2wl}{28} = \frac{32wl}{28} = R_D \text{ by symmetry.}$$

By balance,

$$R_C = \frac{26wl}{28}.$$

The S.F. and B.M. diagrams are shown in fig. 146.

Example 4.

A continuous beam covers three consecutive spans of 25, 30 and 20 feet and carries loads of 2, $1\frac{1}{2}$ and 3 tons per foot run respectively on the three spans. Find the bending moments and pressures at the supports.

Applying the theorem of three moments to spans *AB-BC*,

$$M_A \times 25 + 2M_B \times 55 + M_C \times 30 = \frac{2 \times 25^3}{4} + \frac{1{\cdot}5 \times 30^3}{4}$$

$$\therefore \quad 110M_B + 30M_C = 7812{\cdot}5 + 10125$$
$$= 17937{\cdot}5, \text{ since } M_A = 0 \ldots\ldots\ldots\ldots (1)$$

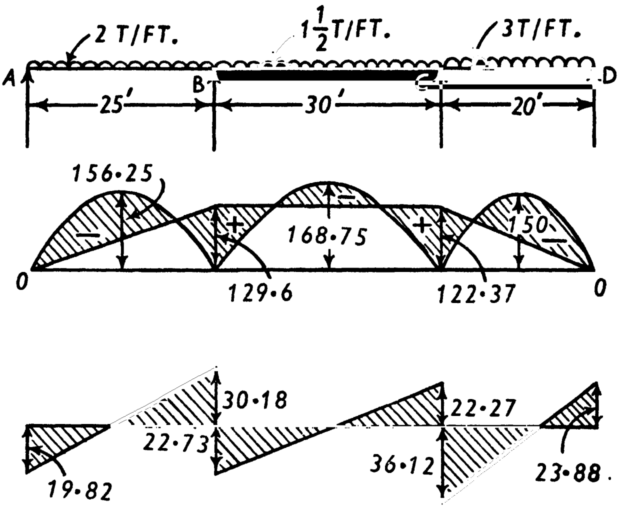


FIG. 147

Considering the spans *BC-CD*,

$$M_B \times 30 + 2M_C \times 50 + M_D \times 20 = \frac{1{\cdot}5 \times 30^3}{4} + \frac{3 \times 20^3}{4}$$

$$\therefore \quad 30M_B + 100M_C = 10125 + 6000$$
$$= 16125, \text{ since } M_D = 0 \ldots\ldots\ldots\ldots (2)$$

From (1) and (2),

$$M_B\,(3{\cdot}67 - {\cdot}3) = 597{\cdot}92 - 161{\cdot}25 = 436{\cdot}67$$

$$\therefore \quad M_B = \frac{436{\cdot}67}{3{\cdot}37} = 129{\cdot}6 \text{ ton-feet,}$$

and $M_C = 122{\cdot}37$ ton-feet.

For reactions at supports,

$$-R_A \times 25 + 50 \times \frac{25}{2} = M_B = +\,129{\cdot}6$$

$$\therefore \quad -R_A + 25 = 5{\cdot}18$$

$$\therefore \quad R_A = 19{\cdot}82 \text{ tons.}$$

Also,

$$-R_A \times 55 - R_B \times 30 + 50 \times 42{\cdot}5 + 45 \times 15 = M_C = +\ 122{\cdot}37$$

$$\therefore\quad -1090{\cdot}1 - 30R_B + 2125 + 675 = 122{\cdot}37$$

$$\therefore\quad -36{\cdot}34 - R_B + 89{\cdot}25 = 0$$

$$\therefore\quad R_B = 52{\cdot}91 \text{ tons.}$$

And finally,

$$-R_D \times 20 + 60 \times 10 = M_C = 122{\cdot}37$$

$$\therefore\quad -R_D + 30 = 6{\cdot}12$$

$$\therefore\quad R_D = 23{\cdot}88 \text{ tons.}$$

By balance, $R_C = 58{\cdot}39$ tons.

The S.F. and B.M. diagrams are shown in fig. 147.

II. Theorem of three moments — any loading.

Consider two consecutive spans AB, BC of a continuous beam carrying a given system of loading. Construct the μ-diagrams for AB, BC treating them as discontinuous freely supported beams. To obtain the relation between the support moments, M_A, M_B and M_C, consider the span AB taking the origin at A and x positive to the right.

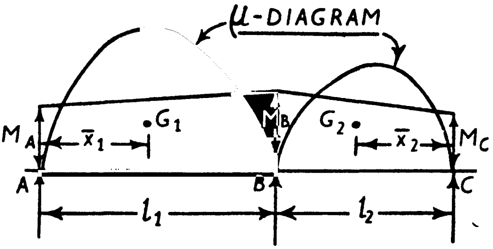


FIG. 148

Now, $EI\dfrac{d^2y}{dx^2} = \mu + \mu'$. Multiply both sides by x and integrate over the whole range from A to B.

$$\therefore\quad \text{we have, } EI\left[x\frac{dy}{dx} - y\right]_0^{l_1} = \int_0^{l_1} \mu.x.dx + \int_0^{l_1} \mu'.x.dx$$

$\therefore\quad EIl_1.i_B = a_1\bar{x}_1 + a'_1\bar{x}'_1$ where a_1 is the area of the μ-diagram and $\bar{x}_1$ the distance of its centre of gravity from A, while $a'_1 \bar{x}'_1$ is the moment of the μ'-diagram about A which is $(M_A + 2M_B) \frac{l_1^2}{6}$

$$\therefore \quad EIl_1 i_B = (M_A + 2M_B) \frac{l_1^2}{6} + a_1 \bar{x}_1$$

$$\text{or} \quad EIi_B = (M_A + 2M_B) \frac{l_1}{6} + \frac{a_1 \bar{x}_1}{l_1} \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Now consider the span BC with the origin at C and x positive to the left. We shall similarly obtain,

$$EIi'_B = (M_C + 2M_B) \frac{l_2}{6} + \frac{a_2 \bar{x}_2}{l_2} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

where a_2 is the area of the μ diagram and $\bar{x}_2$ the distance of its centre of gravity from C.

But $i'_B = -i_B$. Adding (1) and (2), we get,

$$M_A.l_1 + 2M_B(l_1 + l_2) + M_C.l_2 + \frac{6a_1\bar{x}_1}{l_1} + \frac{6a_2\bar{x}_2}{l_2} = 0.$$

It must be remembered that a_1 and a_2 being areas of μ-diagrams are negative according to the convention adopted by us. This is the general theorem of three moments, applicable to beams of uniform section under any system of loading.

The case of uniformly distributed loads w_1 and w_2 over the spans l_1 and l_2 is a special case of this general theorem. For uniformly distributed loading,

$$a_1 = -\tfrac{2}{3} \times \frac{w_1 l_1^2}{8} \times l_1 = -\frac{w_1 l_1^3}{12} \text{ and } \bar{x}_1 = \frac{l_1}{2}$$

$$a_2 = -\tfrac{2}{3} \times \frac{w_2 l_2^2}{8} \times l_2 = -\frac{w_2 l_2^3}{12} \text{ and } \bar{x}_2 = \frac{l_2}{2}$$

The equation therefore reduces to

$$M_A.l_1 + 2M_B(l_1 + l_2) + M_C.l_2 - \frac{6w_1 l_1^4}{24 l_1} - \frac{6w_2 l_2^4}{24 l_2} = 0$$

or $M_A l_1 + 2M_B(l_1 + l_2) + M_C.l_2 = \frac{w_1 l_1^3}{4} + \frac{w_2 l_2^3}{4}$, as already proved.

Example 5.

A continuous beam ABCD, 30 feet long rests on supports A, B, C and D all at the same level. AB = 12 feet; BC = 10 feet and CD = 8 feet. It carries two concentrated loads of 9 tons and 8 tons at 4 feet and 16 feet from A and a uniformly distributed load of $1\frac{1}{2}$ tons per foot run over CD. Find the moments and reactions at the supports.

Applying the theorem of three moments to spans *AB-BC*,

$$M_A \times 12 + 2M_B \times 22 + M_C \times 10 + \frac{6a_1\bar{x}_1}{12} + \frac{6a_2\bar{x}_2}{10} = 0.$$

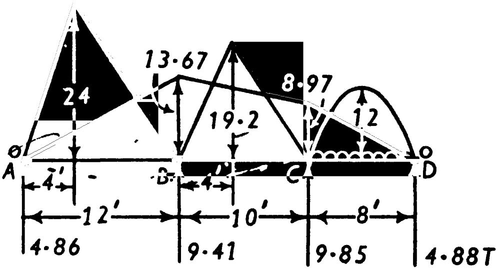


FIG. 149

Now, $a_1\bar{x}_1 = -\{\frac{1}{2} \times 4 \times 24 \times \frac{2}{3} \times 4 + \frac{1}{2} \times 8 \times 24\ (4 + \frac{8}{3})\}$

$= -24\ [\frac{16}{3} +.\frac{80}{3}] = -24 \times 32 = -768.$

And $a_2\bar{x}_2 = -\{\frac{1}{2} \times 6 \times 19{\cdot}2 \times \frac{2}{3} \times 6 + \frac{1}{2} \times 4 \times 19{\cdot}2\ (6 + \frac{1}{3} \times 4)\}$

$= -19{\cdot}2\ \{12 + \frac{44}{3}\} = -19{\cdot}2 \times \frac{80}{3} = -512.$

$$\therefore\quad 44M_B + 10M_C = \frac{6 \times 768}{12} + \frac{6 \times 512}{10} = 691{\cdot}2 \ldots\ldots\ldots (1)$$ ✓

since $M_A = 0$.

Applying the theorem to spans *BC-CD*,

$$10M_B + 2M_C \times 18 + M_D \times 8 + \frac{6a_2\bar{x}_1}{10} + \frac{6a_2\bar{x}_2}{8} = 0.$$

In this case,

$a_1\bar{x}_1 = -\{\frac{1}{2} \times 4 \times 19{\cdot}2 \times \frac{2}{3} \times 4 + \frac{1}{2} \times 6 \times 19{\cdot}2 \times (4 + \frac{1}{3} \times 6)\}$

$= -19{\cdot}2\ \{\frac{16}{3} + 18\} = -19{\cdot}2 \times \frac{70}{3} = -448.$

And $a_2\bar{x}_2 = -\frac{2}{3} \times 12 \times 8 \times 4 = -256.$

$$\therefore \quad 10M_B + 36M_C = \frac{6 \times 448}{10} + \frac{6 \times 256}{8}$$

$$= 268{\cdot}8 + 192 = 460{\cdot}8 \quad \ldots\ldots\ldots\ldots\ldots (2)$$

since $M_D = 0$.

From (1) and (2),

$$4{\cdot}4M_B + M_C = 69{\cdot}12$$

and $${\cdot}28M_B + M_C = 12{\cdot}80$$

$$\therefore \quad 4{\cdot}12M_B = 56{\cdot}32$$

$$\therefore \quad M_B = 13{\cdot}67 \text{ ton-feet.}$$

and $$M_C = 8{\cdot}97 \text{ ton-feet.}$$

For reactions at the supports,

$$-R_A \times 12 + 9 \times 8 = M_B = +13{\cdot}67$$

$$\therefore \quad -R_A + 6 = +1{\cdot}14$$

$$\therefore \quad R_A = 4{\cdot}86 \text{ tons.}$$

Also, $-R_A \times 22 - R_B \times 10 + 9 \times 18 + 8 \times 6 = M_C = +8{\cdot}97$

$$\therefore \quad 22R_A + 10R_B = 201{\cdot}03$$

$$\therefore \quad 2{\cdot}2 \times 4{\cdot}86 + R_B = 20{\cdot}103$$

$$\therefore \quad R_B = 9{\cdot}41 \text{ tons.}$$

Also,

$$-R_D \times 8 + 12 \times 4 = M_C = +8{\cdot}97$$

$$\therefore \quad R_D = 6 - 1{\cdot}12 = 4{\cdot}88 \text{ tons}$$

By balance, $R_C = 9{\cdot}85$ tons.

9. Supports not at the same level: Referring to fig. 148, let the support *B* be δ_1 below *A* and δ_2 below *C*.

Then for the span *AB*, with the origin at *A* and *x* positive to the right,

$$EI\left\{ x\frac{dy}{dx} - y \right\}_0^{l_1} = EI\,(l_1 i_B - \delta_1) = \int_0^{l_1} \mu.x.dx + \int_0^{l_1} \mu'.x.dx$$

$$= a_1\bar{x}_1 + a'_1\bar{x}'_1$$

or $$EIl_1 i_B - EI\ \delta_1 = a_1\bar{x}_1 + (M_A + 2M_B)\,\frac{l_1^2}{6}$$

$$\therefore \quad 6EIi_B = l_1\ (M_A + 2M_B) + \frac{6a_1\bar{x}_1}{l_1} + \frac{6EI\delta_1}{l_1} \ \ldots . (1)$$

Similarly, for the span *BC*, with the origin at *C* and *x* positive to the left,

$$6EIi'_B = l_2(M_C + 2M_B) + \frac{6a_2\bar{x}_2}{l_2} + \frac{6EI\delta_2}{l_2} \quad \ldots\ldots\ldots\ldots (2)$$

Adding (1) and (2), since $i'_B = -i_B$, we get,

$$M_A.l_1 + 2M_B\ (l_1 + l_2) + M_C.l_2 + \frac{6a_1\bar{x}_1}{l_1} + \frac{6a_2\bar{x}_2}{l_2}$$

$$+ 6EI\left(\frac{\delta_1}{l_1} + \frac{\delta_2}{l_2}\right) = 0.$$

Example 6.

In example 2, the support B sinks by $\frac{1}{10}''$ *below A and C. Calculate the support-moments if I for the section* = *221 in*^{4}. *and E* = *13000 tons/in*^{2}.

Considering the spans *AB-BC*,

$\delta_1 = +\frac{1}{10}'' = \delta_2.$

The equation now becomes,

$$44M_B + 10M_C - 691{\cdot}2 + \frac{6\times13000\times221}{144} \times \frac{1}{10}\left(\frac{1}{144} + \frac{1}{120}\right) = 0$$

$$\therefore\quad 44M_B + 10M_C - 691{\cdot}2 + 182{\cdot}8 = 0$$

or $4{\cdot}4M_B + M_C = 50{\cdot}84.$

For the spans *BC-CD*,

$\delta_1 = -\frac{1}{10}''; \ \delta_2 = 0.$

The equation connecting M_B and M_C becomes,

$$10M_B + 36M_C - 460{\cdot}8 + \frac{6\times13000\times221}{144}\left(-\frac{1}{1200}\right) = 0$$

$$\therefore\quad 10M_B + 36M_C - 460{\cdot}8 - 99{\cdot}77 = 0$$

$$\therefore\quad 10M_B + 36M_C = 560{\cdot}57$$

or ${\cdot}28M_B + M_C = 15{\cdot}57.$

Solving for M_B

$4{\cdot}12M_B = 35{\cdot}27$

$\therefore\quad M_B = 8{\cdot}56$ ton-feet

and $\quad M_C = 13{\cdot}18$ ton-feet.

The reactions at the supports may now be worked out as usual. Care must be taken about units in the numerical work involved in continuous beams on supports at different levels.

10. Fixed ends: If the ends are fixed, the necessary equations can be easily obtained. Referring to fig. 148, take the origin at *B* for the span *AB* and *x* positive to the left,

$$EI\frac{d^2y}{dx^2} = \mu + \mu'.$$

Multiplying by *x* and integrating from *B* to *A*, we have

$$EI\left\{x\frac{dy}{dx} - y\right\}_0^{l_1} = \int_0^{l_1} \mu.x.dx + \int_0^{l_1} \mu'.x.dx$$

$$= \text{Moment of the } \mu\text{-diagram about } B$$
$$+ \text{Moment of the } \mu'\text{-diagram about } B$$

$$\therefore \quad 0 = a_1\ (l_1 - \bar{x}_1) + \frac{l_1^2}{6}(M_B + 2M_A)$$

$$\text{or} \quad 2M_A.l_1 + M_B.l_1 + \frac{6a_1\ (l_1 - \bar{x}_1)}{l_1} = 0.$$

This relation can be derived by imagining a zero-span to the left of *A* and applying the theorem of three moments, as usual.

Thus,

$$0 + 2M_A(0 + l_1) + M_B \times l_1 + \frac{6 \times \text{moment of } a_1 \text{ about } B}{l_1} = 0$$

$$\text{i.e. } 2M_A.l_1 + M_B \times l_1 + \frac{6a_1\ (l_1 - \bar{x}_1)}{l_1} = 0$$

$$\text{or} \quad 2M_A + M_B + \frac{6a_1\ (l_1 - \bar{x}_1)}{l_1^2} = 0.$$

If the other end is also fixed, an extra equation can be similarly obtained by imagining a zero-span to the right of the fixed end and applying the theorem of three moments.

Example 7.

In example 5, the end A is fixed while D is free. Calculate the support-moments.

The equations available to evaluate M_A, M_B and M_C are:

$12M_A + 44M_B + 10M_C = 691{\cdot}2$ .... (1)

$10M_B + 36M_C = 460{\cdot}8$ .... (2)

Imagining a zero-span to the left of A,

$$2M_A \times 12 + 12M_B$$

$$= \tfrac{6}{12}\left[\tfrac{1}{2} \times 24 \times 8 \times \tfrac{16}{3} + \tfrac{1}{2} \times 24 \times 4 \times \left(8 + \tfrac{4}{3}\right)\right]$$

$$= 480$$

$\therefore \quad 2M_A + M_B = 40$ .... (3)

Eliminating M_A between (1) and (3),

$$38M_B + 10M_C = 451{\cdot}2$$

$$\therefore \quad 3{\cdot}8M_B + M_C = 45{\cdot}12.$$

From (2), $\cdot 28M_B + M_C = 12{\cdot}80$

$$\therefore \quad 3{\cdot}52M_B = 32{\cdot}32$$

$$\therefore \quad M_B = 9{\cdot}18 \text{ ton-feet}$$

$$M_C = 10{\cdot}23 \text{ ton-feet}$$

and $M_A = 15{\cdot}41$ ton-feet.

11. Slope-deflection method: Consider the span *AB* of a continuous beam under a given system of loading. Referring to fig. 148, we have seen that,

$EIi_B.l_1 = a_1\bar{x}_1 + \dfrac{(M_A + 2M_B)\, l_1^2}{6}$, where $a_1\bar{x}_1$ is the moment of the μ-diagram about *A* and M_A and M_B are the support-moments.

Similarly, $-EIi_A.l_1 = a_1\,(l_1 - \bar{x}_1) + \dfrac{(M_B + 2M_A)\, l_1^2}{6}$

These expressions are easily remembered by the analogy to the equations for reactions at the supports of a loaded beam. To obtain the reactions, we take moments of the system about a support, say *A*, and we get,

$R_B \times l_1 =$ moment of the load system about A. Similarly, to obtain the slope at B, we treat the B.M. diagram as if it was a load system and taking moments about A, we get,

$$EIi_B \times l_1 = \text{moment of the B.M. diagram about } A$$

$$= a_1\bar{x}_1 + \frac{l_1^2}{6}(M_A + 2M_B).$$

Similarly, taking moments of the B.M. diagram about B we have,

$$-EIi_A \times l_1 = a_1(l_1 - \bar{x}_1) + \frac{l_1^2}{6}(M_B + 2M_A).$$

The relations between the end-slopes and the support-moments in a beam AB, of span l, may therefore be stated as under:

$$i_B = \frac{a\bar{x}}{EIl} + \frac{(M_A + 2M_B)\,l}{6EI} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (A)$$

$$-i_A = \frac{a\,(l - \bar{x})}{EIl} + \frac{(M_B + 2M_A)\,l}{6EI} \quad \ldots\ldots\ldots\ldots\ldots\ldots (B)$$

Expressing M_A and M_B in terms of i_A and i_B,

$$M_B = \frac{2a}{l} - \frac{6a\bar{x}}{l^2} + \frac{2EI}{l}(2i_B + i_A) \quad \ldots\ldots\ldots\ldots\ldots\ldots (C)$$

$$\text{and } M_A = -\frac{4a}{l} + \frac{6a\bar{x}}{l^2} - \frac{2EI}{l}(2i_A + i_B) \quad \ldots\ldots\ldots\ldots\ldots (D)$$

These four expressions are known as slope-deflection equations and form the basis of the slope-deflection method for solving problems on framed structures. We shall illustrate the use of this method on problems on continuous beams. We shall resolve questions 4 and 5 of this chapter by the slope-deflection method.

Referring to fig. 147, consider the span AB and take moments about A of the B.M. diagram. We get,

$$EIi_B \times 25 = -\tfrac{625}{4} \times \tfrac{2}{3} \times 25 \times \tfrac{25}{2} + M_B \times \tfrac{1}{2} \times 25 \times \tfrac{2}{3} \times 25$$

$$\therefore \quad EIi_B = -\frac{625 \times 25}{12} + \frac{25}{3}M_B \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

For the span *BC*,

$$EIi_C \times 30 = -\tfrac{675}{4} \times \tfrac{2}{3} \times 30 \times 15 + (M_B + 2M_C)\,\tfrac{900}{6}$$

and $-EIi_B \times 30 = -\tfrac{675}{4} \times \tfrac{2}{3} \times 30 \times 15 + (M_C + 2M_B)\,\tfrac{900}{6}$

$$\therefore \qquad EIi_B = 1687{\cdot}5 - 5\,(2M_B + 2M_C) \;\dots\dots\dots\dots (2)$$

and $\qquad EIi_C = -\,1687{\cdot}5 + 5\,(M_B + 2M_C) \;\dots\dots\dots\dots (3)$

Now consider the span *CD*; taking moments about *D*,

$$-EIi_C \times 20 = -150 \times \tfrac{2}{3} \times 20 \times 10 + M_C \times \tfrac{1}{2} \times 20 \times \tfrac{2}{3} \times 20$$

$$\therefore \qquad EIi_C = 1000 - \tfrac{20}{3}\,M_C \;\dots\dots\dots\dots (4)$$

Equating (1) and (2),

$$-\,625 \times 25 + 100M_B = 1687{\cdot}5 \times 12 - 60\;(2M_B + M_C)$$

$$\therefore \quad 220M_B + 60M_C = 20250 + 15625 = 35875$$

$$\therefore \quad 3{\cdot}67M_B + M_C = 597{\cdot}92 \;\dots\dots\dots\dots (5)$$

Equating (3) and (4),

$$-\,1687{\cdot}5 + 5\;(M_B + 2M_C) = 1000 - \tfrac{20}{3}\,M_C$$

$$\therefore \quad -\,5062{\cdot}5 + 15\;(M_B + 2M_C) = 3000 - 20M_C$$

$$\therefore \quad 15M_B + 50M_C = 8062{\cdot}5$$

$$\therefore \quad {\cdot}3M_B + M_C = 161{\cdot}25 \;\dots\dots\dots\dots (6)$$

Solving for M_B and M_C from (5) and (6),

$$3{\cdot}37M_B = 436{\cdot}67$$

or $\qquad M_B = 129{\cdot}6$ ton-feet

and $\qquad M_C = 122{\cdot}37$ ton-feet, as before.

To solve question 5 by this method, refer to fig. 149. For the span *AB*, taking moments of the B.M. diagram about *A*,

$$\begin{aligned} EIi_B \times 12 &= -\{24 \times \tfrac{4}{2} \times \tfrac{2}{3} \times 4 + 24 \times \tfrac{8}{2}\,(4 + \tfrac{8}{3})\} \\ &\qquad + M_B \times \tfrac{12}{2} \times \tfrac{2}{3} \times 12 \\ &= -\,24 \times 32 + 48M_B \end{aligned}$$

$$\therefore \qquad EIi_B = -\,64 + 4M_B \;\dots\dots\dots\dots (7)$$

For the span *BC*,

$$\begin{aligned} EIi_C \times 10 &= -\{19{\cdot}2 \times \tfrac{4}{2} \times \tfrac{2}{3} \times 4 + 19{\cdot}2 \times \tfrac{6}{2} \times (4 + \tfrac{1}{3} \times 6)\} \\ &\qquad + \tfrac{100}{6}\,(M_B + 2M_C) \\ &= -\,19{\cdot}2 \times \tfrac{70}{3} + (M_B + 2M_C)\,\tfrac{100}{6} \end{aligned}$$

$$\therefore \qquad EIi_C = -\,44{\cdot}8 + \tfrac{5}{3}(M_B + 2M_C) \;\dots\dots\dots\dots (8)$$

And

$$-EIi_B \times 10 = -\{19\cdot2 \times \tfrac{6}{2} \times \tfrac{2}{3} \times 6 + 19\cdot2 \times \tfrac{4}{2} \times (6 + \tfrac{4}{3})\}$$

$$+ \tfrac{100}{6}(2M_B + M_C)$$

$$= -19\cdot2 \times \tfrac{80}{3} + \tfrac{100}{6}(2M_B + M_C)$$

$$\therefore \quad EIi_B = 51\cdot2 - \tfrac{5}{3}(2M_B + M_C) \ldots\ldots\ldots\ldots (9)$$

Finally, for he span *CD*, taking moments about *D*,

$$-EIi_C \times 8 = -\tfrac{2}{3} \times 12 \times 8 \times 4 + M_C \times \tfrac{8}{2} \times \tfrac{2}{3} \times 8$$

$$\therefore \quad EIi_C = 32 - \tfrac{8}{3}M_C \ldots\ldots\ldots\ldots (10)$$

Equating (7) and (9),

$$-64 + 4M_B = 51\cdot2 - \tfrac{5}{3}(2M_B + M_C)$$

$$\therefore \quad -192 + 12M_B = 153\cdot6 - 10M_B - 5M_C$$

$$\therefore \quad 22M_B + 5M_C = 345\cdot6$$

$$\therefore \quad 4\cdot4M_B + M_C = 69\cdot12 \ldots\ldots\ldots\ldots (11)$$

Equating (8) and (10),

$$-44\cdot8 + \tfrac{5}{3}(M_B + 2M_C) = 32 - \tfrac{8}{3}M_C$$

$$\therefore \quad -134\cdot4 + 5M_B + 10M_C = 96 - 8M_C$$

$$\therefore \quad 5M_B + 18M_C = 230\cdot4$$

$$\therefore \quad \cdot28M_B + M_C = 12\cdot8 \ldots\ldots\ldots\ldots (12)$$

From (11) and (12),

$$4\cdot12M_B = 56\cdot32$$

$$\therefore \quad M_B = 13\cdot67 \text{ ton-feet}$$

and $M_C = 8\cdot97$ ton-feet, as before.

12. Uneven supports: If the support at *B* is δ lower than the support *A*, in the span *AB* of a continuous beam, the expression for i_B will now be,

$$EI\,(li_B - \delta) = a\bar{x} + \frac{l^2}{6}(M_A + 2M_B)$$

$$\therefore \quad i_B = \frac{a\bar{x}}{EIl} + \frac{\delta}{l} + \frac{(M_A + 2M_B)\,l}{6EI} \ldots\ldots\ldots\ldots (13)$$

Similarly,

$$-i_A = \frac{a\,(l - \bar{x})}{EI.l} - \frac{\delta}{l} + \frac{(M_B + 2M_A)\,l}{6EI} \ldots\ldots\ldots\ldots (14)$$

Expressing M_A and M_B in terms of i_A and i_B,

$$M_A = -\left\{\frac{2EI}{l}\left(2i_A + i_B - \frac{3\delta}{l}\right) + \frac{2a}{l}\left(2 - \frac{3\bar{x}}{l}\right)\right\} \quad \ldots\ldots(15)$$

and $$M_B = \left\{\frac{2EI}{l}\left(i_A + 2i_B - \frac{3\delta}{l}\right) + \frac{2a}{l}\left(1 - \frac{3\bar{x}}{l}\right)\right\} \quad \ldots\ldots\ldots(16)$$

Expressions 13, 14, 15 and 16 are slope-deflection equations in the most general forms. Their use in a practical form will be dealt with later.

13. Method of Moment Distribution: This elegant method due to Prof. Hardy Cross is so important and of such wide application to framed structures that it will be treated in a separate chapter.

EXAMPLES VII

(1) A horizontal beam, 16 feet span, is built-in at one end and the other end rests freely on a support. It carries a concentrated load of 5 tons at 4 feet from the fixed end. Determine the reaction at the support and the bending moment at the fixed end, using Macaulay's method. Draw the S.F. and B.M. diagrams for the beam. (·43 ton; 13·12 ton-feet.)

(2) A horizontal beam *AB*, 20 feet span, is built-in at *A* and rests freely on a support at the other end *B*. It carries a uniformly distributed load of 1 ton per foot run over the whole length and also a concentrated load of 20 tons at the middle. Use Macaulay's method for the fixing moment at *A* and the reaction at *B*. Draw the S.F. and B.M. diagrams. (125 ton-feet; 13·75 tons.)

(3) A beam, 24 feet span, has its ends built-in and carries a uniformly distributed load of 300 lb. per foot run. Find the maximum B.M. and construct the B.M. diagram. (14400 lb-feet.)

(4) A horizontal beam of span l is fixed at both ends. Two equal loads W each, are placed at equal distances a from the ends. Prove that the bending moment at the centre is $\frac{Wa^2}{l}$ and that the deflections under either load and at the centre are $\frac{Wa^3\,(2l-3a)}{6EI.l}$ and $\frac{Wa^2\,(3l-4a)}{24EI}$ respectively.

(5) A built-in beam of 20 feet span, carries a load of 8 tons at the centre and loads of 5 tons at 3 feet from each end. Calculate the maximum bending moments and the positions of the points of contraflexure. (+32·75 tons-ft.; —22·25 ton-ft.; 4·4375 ft. from the ends.)

(6) A beam, 24 feet span, is built-in at both ends and carries a uniformly distributed load of $1\frac{1}{2}$ tons per foot run. It also carries two point-loads of 15 tons each at 6 feet from the ends. Draw the B.M. diagram and state the maximum values. State the position of points of contraflexure. (+139·5 ton-feet;—58·5 ton-feet; 4·74 feet from ends.)

(7) A built-in beam of 20 feet span carries a uniformly distributed load of $\frac{3}{4}$ ton per foot run from one end to the centre of the span. Find the fixing couples at the ends. (7·81 ton-ft.; 17·19 ton-ft.)

(8) A built-in beam of 18 feet span carries a load of 8 tons at 6 feet from one end. Find the position and amount of maximum deflection, if I for the section = 221 in^4. and E = 13000 tons/in^2.
(·113″ at 10·29′ from the other end.)

(9) A built-in beam of 20 feet span carries two loads of 6 tons and 12 tons at 4 feet and 15 feet from the left hand end. Find the fixing couples at the ends. (26·61 ton-ft.; 37·59 ton-ft.)

(10) A beam of span l is fixed at both ends. A couple μ is applied to the beam at the middle about a horizontal axis at right angles to the beam. Prove that the fixing couple at each support is $\frac{\mu}{4}$ in the same direction and that the slope at the centre is $\frac{\mu l}{16EI}$.
(C. U.)

(11) A beam of span l carries a central load W. It is so constrained at the ends that when the end-slope is i, the restraining couple at the supports is $\mu.i$. Prove that the magnitude of the restraining couple at each end is $\frac{\mu W.l^2}{8}-(\mu.l+2EI)$ and that the magnitude of the central deflection is $\frac{Wl^3}{192EI}\left(\frac{\mu.l+8EI}{\mu.l+2EI}\right)$. (C. U.)

(12) A beam of 20 feet span is built-in at both ends. When a uniformly distributed load of $\frac{3}{4}$ ton per foot run is placed on the beam, the level of the right-hand support sinks $\frac{1}{2}$″ below that of the left hand one. Find the magnitudes of the support-moments, given I for the section = 221 in^4. and E = 13000 tons/in^2.

(37·5 ton-ft. at left; 12·5 ton-ft. at the right-hand support.)

(13) A beam 45 feet long is freely supported at the ends. It carries an intermediate support at the same level as the ends dividing it into two spans of 25 and 20 feet. On the longer span there is a

uniformly distributed load of $\frac{1}{2}$ ton per foot run and on the other, the rate of loading is $\frac{3}{4}$ ton per foot run. Calculate the moment at the central support and the reactions at all the supports.

(38·37 ton-feet; 4·72 tons; 17·2 tons; 5·58 tons.)

(14) A girder 100 feet long rests on five supports at the same level forming four equal spans of 25 feet each. If it carries a uniformly distributed load of $1\frac{1}{2}$ tons per foot run, calculate the reactions and moments at the supports.

$$\left\{\begin{array}{lllll} 14{\cdot}73; & 42{\cdot}86; & 34{\cdot}82; & 42{\cdot}86; & 14{\cdot}73 \text{ tons} \\ 0; & 100{\cdot}45; & 67; & 100{\cdot}45; & 0 \text{ ton-feet.} \end{array}\right\}$$

(15) A girder 50 feet long, carrying a uniformly distributed load of 2 tons per foot run, covers three spans, the end spans being 15 feet each and the central one 20 feet. Calculate the maximum bending moments and the positions of the points of contraflexure. (+63·20 ton-feet; −36·80 ton-feet; 10·79 and 18·93 feet from ends.)

(16) For the girder in the above question if the loading consists of 2 tons per foot run for the central and one outer span and $1\frac{1}{2}$ tons per foot run on the remaining span, calculate the reactions at the supports. (10·66; 39·76; 34·60; 7·48 tons.)

(17) A girder 160 feet long is continuous over three spans. The two end spans are each 50 feet long, the central one being 60 feet long. The loading on the spans is 1, 2 and 3 tons per foot run respectively from left to right. Determine the moments and reactions at the supports.

(0; 413·1; 804·24; 0 ton-feet; 16·74; 86·74; 157·6; 58·92 tons.)

(18) A girder 50 feet long is supported at the ends and has an intermediate support at 30 feet from one end. It carries a concentrated load of 12 tons at the middle of each span. Find the reactions at the supports. (4·05; 16·875; 3·075 tons.)

(19) In the above example, if the central prop is $\frac{1}{2}''$ lower than the end supports, calculate the reactions, given I for the section $=$ 725 in^4. and $E = 13000$ tons/in^2. (4·50; 15·74; 3·76 tons.)

(20) A beam *ABCD*, 40 feet long is continuous over three spans. $AB = 16$ feet; $BC = 12$ feet and $CD = 12$ feet; the supports being at the same level. There is a uniformly distributed load of $1\frac{1}{2}$ tons per foot run over *BC*. On *AB*, there is a point-load of 15 tons at 4 feet from *A* and on *CD*, there is a point-load of 20 tons at 9 feet from *D*. Calculate the moments and reactions at the supports.

$$\left\{\begin{array}{llll} 0; & 21{\cdot}71; & 27{\cdot}76; & 0 \text{ ton-feet,} \\ 9{\cdot}89; & 13{\cdot}63; & 26{\cdot}79; & 2{\cdot}69 \text{ tons.} \end{array}\right\}$$

(21) In the above example, if the support B sinks $\frac{1}{4}''$, calculate the reactions at the supports, I for the section being 375 in^4. and $E = 13000$ tons/in^2. (10·72; 10·8; 29·68; 1·8 tons)

(22) In example 20, if the ends A and D are both built-in, calculate the support-moments.

(33·10; 12·55; 26·31; 14·97 ton-feet.)

(23) A beam $ABCD$, 28 feet long is continuous over three spans of 10, 10 and 8 feet. The ends are freely supported and all the supports are at the same level. There is a uniformly distributed load of $1\frac{1}{2}$ tons per foot run over each of the 10 ft. spans and a load of 2 tons per foot run over the 8 feet-span. Use the slope-deflection method to evaluate the support-moments.

(0; 15·44; 13·24; 0 ton-feet.)

(24) In example 23, if the support B sinks by $\frac{1}{8}''$, calculate the support-moments, using the slope-deflection equations. I for the beam section $= 145{\cdot}6$ in^4.; $E = 13000$ tons/in^2.

(0; 10·41; 16·91; 0 ton-feet.)

(25) A beam AB of uniform section, simply supported at its ends distant $4a$ apart is propped at the mid-point C to the same level as the ends. A concentrated load W is applied at the mid-point of AC and a concentrated load nW at the mid-point of CB. Show that by a suitable choice for the value of n, it is possible to arrange that the slope of the beam at the end A is zero. Assuming this value for n, determine the slopes at B and C in terms of W and a and the flexural rigidity EI. (L. U.)

$$\left\{n = 3;\ EIi_B = -\frac{Wa^2}{2};\ EIi_C = \frac{Wa^2}{4}\right\}$$

(26) A continuous beam ABC has two spans $AB = l_1$ and $BC = l_2$. The beam simply rests on the end supports and it carries a uniformly distributed load of w per unit length on its whole length. If the support B sinks an amount δ below the level of the supports A and C, show that the reaction at B is

$$R_B = \frac{w\,(l_1 + l_2)}{2} + \frac{w\,(l_1^3 + l_2^3)}{8\,l_1 l_2} - \frac{3EI\delta\,(l_1 + l_2)}{l_1^2\,l_2^2} \qquad \text{(L. U.)}$$

(27) A beam AB of uniform section, is positioned-fixed and direction-fixed at both ends of a span of 24 feet. At a point 15 feet from the left hand end, a connection made to the beam exerts a vertically downward force of 8 tons and also a clockwise couple of 20 ton-feet in a vertical plane of bending; in addition, the beam carries a uniformly distributed load of $\frac{1}{4}$ ton per foot run on the whole span.

Determine the fixing moments and reactions at the supports and make dimensioned sketches of the B.M. and S.F. diagrams. (L. U.)

$$\left\{\begin{array}{ll} M_A = 22\frac{5}{16} \text{ ton-ft.}; & M_B = 41\frac{11}{16} \text{ ton-ft.}; \\ R_A = 4\frac{23}{64} \text{ tons}; & R_B = 9\frac{41}{64} \text{ tons.} \end{array}\right\}$$

(28) A beam AB of length l and uniform flexural rigidity EI is fixed horizontally at A and pinned at B at the same level as A. An anticlockwise moment M is applied at B. Neglecting the weight of the beam, find the reaction and fixing moment at A, the slope of the beam at B and the position and magnitude of the maximum deflection.

If B drops an amount δ relative to A, find the new values of the reaction and moment at A and of the slope at B. (L. U.)

$$\left\{\begin{array}{l} \text{(i)} \quad R_A = \frac{3M}{2l};\ M_A = \tfrac{1}{2}M;\ \ -i_B = \frac{Ml}{4EI}; \\ \quad Y_{max} = \frac{Ml^2}{27EI} \text{ at } \frac{2l}{3} \text{ from fixed end.} \\ \text{(ii)} \quad R_A = \frac{3M}{2l} + \frac{3EI\delta}{l^3};\ M_A = \frac{M}{2} + \frac{3EI\delta}{l^2};\ -i_B = \frac{Ml}{4EI} - \frac{3\delta}{2l}. \end{array}\right\}$$

(29) An unloaded horizontal beam ABC of constant flexural rigidity EI and length $2l$, is rigidly fixed to stanchions at each end and simply supported at the mid-point B. The left hand end A remains precisely horizontal but due to the bending of the stanchion at C, that end is rotated through a small anticlockwise angle θ_C. Determine the values of the bending moments and reactions at A, B and C in terms of the given symbols and give dimensioned sketches of the S.F. and B.M. diagrams which result from the change of slope at C. (L. U.)

$$\left\{\begin{array}{lll} M_A = -\frac{EI\theta_C}{2l}; & M_B = +\frac{EI\theta_C}{l}; & M_C = -\frac{7EI\theta_C}{2l} \\ R_A = -\frac{3EI\theta_C}{2l^2}; & R_B = +\frac{6EI\theta_C}{l^2}; & R_C = -\frac{9EI\theta_C}{2l^2} \end{array}\right\}$$

(30) A beam of length 100 inches is built-in at each end and loaded at the centre by a load of 10000 lb. Under load, one end remains fixed horizontally and without vertical movement but the other yields vertically $F \times 10^{-4}$ inch under a vertical load of F lb. and rotates through an angle of $M \times 10^{-7}$ radians under a bending moment of M lb-in. The value of $EI = 650 \times 10^6$ lb-in^2. Sketch the B.M. diagram and determine the fixing moments at each end. (L. U.)

$$\left\{\begin{array}{l} \text{At the fixed end, } + 243784 \text{ lb-inches.} \\ \text{At the other, } - 20720 \text{ lb-inches.} \end{array}\right\}$$

(31) A horizontal beam of 21 feet span for which EI is $1\cdot6 \times 10^6$ ton-in^2. is fixed at each end. The fixings are not rigid but their stiffness is such that the inclination of the beam θ at the ends is given by $\theta = 0\cdot0023\ M$, θ being in degrees and M the fixing moment in ton-inches. Find the fixing moments at the ends of this beam when a vertical load of 7·5 tons acts at 9 feet from one end. Neglect the weight of the beam itself. (L. U.)

(11·69 ton-feet; 13·86 ton-feet.)

(32) An encastré beam of span l carries a load wl uniformly distributed over the span. The second moment of area of the beam section is not the same throughout; for a length $\frac{l}{4}$ at each end, the value is $2I$ and for the middle length $\frac{l}{2}$ it is I. Determine the bending moments at the ends of the beam and sketch the B.M. diagram showing on it the values at the ends and at mid-span. (L. U.)

$$\left\{\begin{array}{l}\text{At each end, } + \dfrac{3wl^2}{32} \\ \text{At centre, } - \dfrac{wl^2}{32}\end{array}\right\}$$

CHAPTER VIII

DIRECT AND BENDING STRESSES

1. Combined bending and direct stresses: Often a structural member has to resist a bending moment as well as direct axial loading which may be tensile or compressive. This happens when the load coming on it is eccentric. Wind pressure on a wall or chimney shaft, water or earth pressure on a wall — as in the case of dams and retaining walls — are also instances in which a section is subjected to direct loading accompanied by a bending moment. In the study of beams we have seen that the nature of stress set up across a section is tensile and compressive, varying from zero at the neutral axis to a maximum at the extremities of the section. For a member subjected to a direct load and a bending moment, the actual stress set up at any point in the resisting section will therefore be the algebraic sum of the stresses due to the direct load and the bending moment.

2. Eccentric loading: Consider a short strut subjected to a longitudinal push P, which is off the geometric axis by e. If we introduce two equal and opposite forces $P - P$, at the axis, the system reduces to:

(i) An axial force P and

(ii) A couple $M = P.e$.

Any normal section of the strut will have to resist these two actions. The resistance to the axial load P is $p_0 = \frac{P}{A}$, where A is the area of the resisting section, and is uniform throughout the section. The resistance to M will, as usual, be tensile or compressive depending upon the position of the point with regard to the neutral axis. For a point in the section at a distance y from the N.A., the bending stress will be $p = \frac{M.y}{I}$ and the total stress at the point will be $(p_0 \pm p)$ or

$\left(\frac{P}{A} \pm \frac{M.y}{I}\right)$, the sign depending upon its position. The maximum bending stress p_b is at the extremities of the section, its value for a symmetrical section being $p_b = \frac{M}{Z} = \frac{P.e}{Z}$ where Z is the modulus of section. The total stress across the section due to both the actions will vary by a straight-line law from $p_{min} = p_o - p_b = \frac{P}{A} - \frac{M}{Z}$, to $p_{max} = p_o + p_b = \frac{P}{A} + \frac{M}{Z}$, as shown in fig. 150.

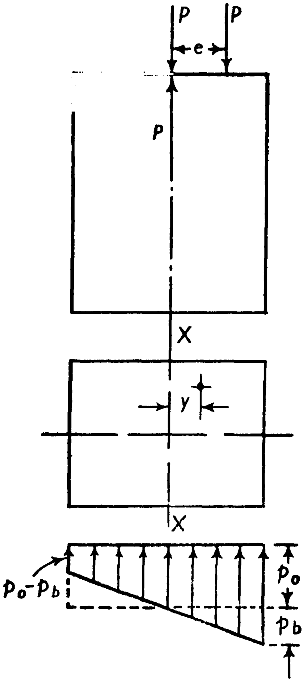


FIG. 150

Whether the stress will be of the same sign throughout the section depends upon the value of p_b. If $p_b \leqq p_o$, the stress will have the same sign as p_o; if, however $p_b > p_o$, the stress will change sign, being partly tensile and partly compressive across the section. The three possible distributions are indicated in fig. 151. It may be noted that when $p_b = p_o$, the value of $p_{max} = 2p_o$.

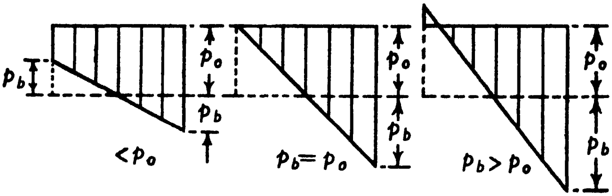


FIG. 151

3. Limit of eccentricity: It is interesting to investigate the limit of eccentricity which may be permitted for different sections so that the stress across the section does not change sign. For the stress to be of the same sign,

$$p_b \leqq p_o$$

$$\text{i.e.} \quad \frac{M}{Z} \leqq \frac{P}{A}$$

$$\therefore \quad \frac{P.e.d}{2I} \leqq \frac{P}{A}$$

$$\therefore \quad \frac{P.e.d}{2A.k^2} \leqq \frac{P}{A}$$

or $e \leqq \dfrac{2k^2}{d}$, where k is the radius of gyration of the section with regard to the N.A. and d is the depth of section.

For a rectangular section of breadth b and depth d, $I = \frac{1}{12}bd^3$, if the load line is in the vertical plane bisecting b.

$$\text{Now,} \ p_o = \frac{P}{A} = \frac{P}{bd}$$

$$\text{and} \quad p_b = \frac{M}{Z} = \frac{6Pe}{bd^2}.$$

$$\text{If} \quad p_b < p_o$$

$$\frac{6P.e}{bd^2} < \frac{P}{bd}$$

$$\text{i.e.} \quad e < \tfrac{1}{6}d.$$

The load can be eccentric on either side of the geometrical axes. Therefore the stress will be of the same sign throughout the section if the load line is within the middle third. If the load line is in the vertical plane bisecting the depth *d* the permissible range of eccentricity will be $\frac{b}{2}$ as shown in fig. 152(*a*).

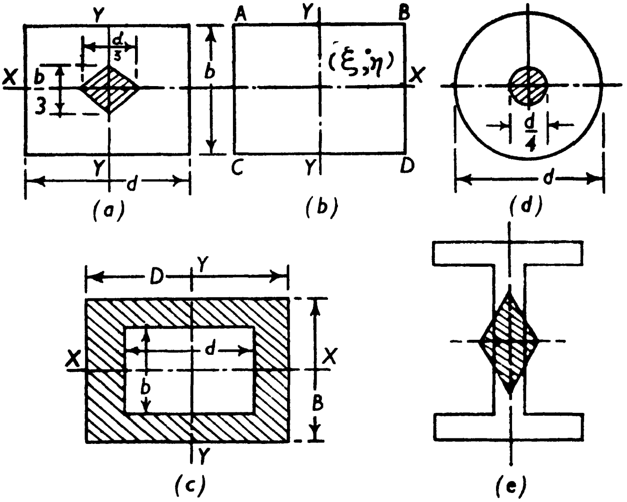


FIG. 152

It is possible that the load line is on neither of the principal axes. It must however lie within the rhombus whose diagonals are $\frac{d}{3}$ and $\frac{b}{3}$ on the respective principal axes. This rhombus is known as the "Core" or "Kernel" of the section. Let (ξ,η) be the coordinates of the point of incidence of the load line on the section, referred to *XX* and *YY* as the axes of reference fig. 152(*b*).

There will thus be a bending moment $M_1 = P.\eta$ under which the section will bend about the axis *XX*, accompanied by a bending moment $M_2 = P.\xi$ under which there will be bending about the axis *YY*. In fig. 152(*b*), the point *C* will have extreme intensities of stress, both tensile, due to the two moments. The bending stress at C due to M_1 and M_2 will be:

$$M_1/\frac{I_{xx}}{b/2} + M_2/\frac{I_{yv}}{d/2}.$$

Since $I_{xx} = \frac{1}{12}db^3$ and $I_{yy} = \frac{1}{12}bd^3$, the bending stress

$$p_b = \frac{6P.\eta}{db^2} + \frac{6P\xi}{bd^2} = \frac{P}{bd}\left(\frac{6\xi}{d} + \frac{6\eta}{b}\right)$$

The total stress at C is, therefore,

$$p_o - p_b = \frac{P}{bd}\left\{1 - \frac{6\xi}{d} - \frac{6\eta}{b}\right\}$$

For the stress to remain of the same sign throughout the section,

$(p_o - p_b)$ must be positive, i.e.,

$$1 - \frac{6\xi}{d} - \frac{6\eta}{b} \geqq 0$$

$$\therefore \quad \frac{6\xi}{d} + \frac{6\eta}{b} \leqq 1.$$

The deviation of the load line is governed by the straight-line $\frac{6\xi}{d} + \frac{6\eta}{b} = 1$, whose intercepts on the axes are respectively $\frac{d}{6}$ and $\frac{b}{6}$. This is true for the load line in the first quadrant. For positions in other quadrants, the limiting lines will be the sides of the "Core" rhombus.

For a hollow rectangular section: Let the dimensions of the section be $B \times D$ external and $b \times d$ internal as in fig. 152(c). If the load is eccentric by e on the axis XX,

$$Z = \frac{BD^3 - bd^3}{6D} \text{ and } p_b = \frac{M}{Z} = \frac{6P.e.D}{BD^3 - bd^3}$$

The limit of eccentricity on either side of the centre is given by,

$$\frac{6P.e.D}{BD^3 - bd^3} \leqq \frac{P}{(BD - bd)}$$

i.e. $$e \leqq \frac{(BD^3 - bd^3)}{6D\,(BD - bd)}$$

Similarly, if the load line is on the axis YY, the limit of eccentricity on either side of the centre will be,

$$\frac{DB^3 - db^3}{6B\,(BD - bd)}$$

For a solid circular section of diameter d,

$$Z = \frac{\pi}{32}d^3 \text{ and } p_b = \frac{M}{Z} = \frac{32\ P.e}{\pi d^3}.$$

The limit of eccentricity is given by,

$$\frac{32\ P.e}{\pi d^3} \leqq p_o$$

$$\leqq \frac{4P}{\pi d^2}$$

i.e. $e \leqq \frac{d}{8}$. The "Core" in this case is a circle with the same centre and diameter $= \frac{d}{4}$. [See fig 152(*d*)].

For a hollow circular section, ext. diam. D, int. diam. d.

$$Z = \frac{\pi\,(D^4 \quad d^4)}{32D} \text{ and } p_b = \frac{32\ P.e.D}{\pi\,(D^4 - d^4)}.$$

The limit of eccentricity will be obtained from

$$\frac{32\ P.e\,D}{\pi\,(D^4 - d^4)} \leqq \frac{4P}{\pi\,(D^2 - d^2)}$$

i.e. $$e \leqq \left(\frac{D^2 + d^2}{8D}\right).$$

The core in this case is a concentric circle of diameter, $\frac{D^2 + d^2}{4D}$.

For an *I*-section, the core can be similarly obtained. [See fig. 152(*e*)].

Example 1.

A steel flat 6″ wide and $\frac{3}{4}$″ thick is subjected to a pull of 18 tons, which is off the geometrical axis by $\frac{1}{8}$ inch in the plane which bisects the thickness. Determine the maximum and minimum stress intensities set up in the section.

$$P = 18 \text{ tons}$$
$$A = 6 \times \tfrac{3}{4} = 4\tfrac{1}{2} \text{ sq. in.}$$
$$M = P.e = 18 \times \tfrac{1}{8} = 2{\cdot}25 \text{ ton-in.}$$
$$Z = \tfrac{1}{6}bd^2 = \tfrac{1}{6} \times \tfrac{3}{4} \times 36 = 4{\cdot}5 \text{ in}^3.$$

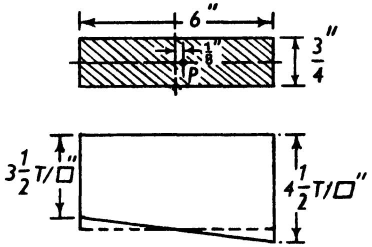


FIG. 153

$$p_o = \frac{P}{A} = \frac{18}{4{\cdot}5} = 4 \text{ tons/in}^2.$$
$$p_b = \frac{M}{Z} = \frac{2{\cdot}25}{4{\cdot}5} = {\cdot}5 \text{ ton/in}^2.$$
$$\therefore\ p_{max} = p_o + p_b = 4 + {\cdot}5 = 4{\cdot}5 \text{ tons/in}^2\text{., tensile.}$$
$$p_{min} = p_o - p_b = 4 - {\cdot}5 = 3{\cdot}5 \text{ tons/in}^2\text{., tensile.}$$

The stress distribution is shown in fig. 153.

Example 2.

A short C.I. column of hollow circular section has a projecting bracket carrying a load of 5 tons. The load line is off the axis of the column by 12 inches. The external diameter of the column is 12" and the thickness of metal is 1". Find the maximum and minimum stress intensities in the section.

$$\text{The area of section } A = \frac{\pi}{4}(12^2 - 10^2)$$
$$= 11\pi \text{ in}^2.$$
$$= 34{\cdot}56 \text{ in}^2.$$
$$\text{Modulus of section } Z = \frac{\pi}{32} \times \frac{244 \times 44}{12}$$
$$= 87{\cdot}84 \text{ in}^3.$$
$$p_o = \frac{P}{A} = \frac{5}{34{\cdot}56} = {\cdot}1447 \text{ ton/in}^2\text{., compressive.}$$

$$p_b = \frac{M}{Z} = \frac{5 \times 12}{87{\cdot}84} = {\cdot}6831 \text{ ton/in}^2.$$

$p_{max} = p_o + p_b = {\cdot}1447 + {\cdot}6831 = {\cdot}8278$ ton/in^2., compressive.
$p_{min} = p_o - p_b = {\cdot}1447 - {\cdot}6831 = -{\cdot}5384$ ton/in^2.,
i.e. ·5384 ton/in^2., tensile.

Example 3.

A mild steel strut of Tee-section 6″ × 4″ × ½″ carries a load of 10 tons in the central plane bisecting the web at 1·5″ from the base. Calculate the maximum and min. stress intensities induced in the section. For the 6″ × 4″ × ½″ Tee-section, $A = 4{\cdot}771$ in^2.; $J = {\cdot}968$″; $I_{xx} = 6{\cdot}07$ in^4.; $I_{yy} = 8{\cdot}64$ in^4.

The eccentricity of loading $e = 1{\cdot}5 - {\cdot}968 = {\cdot}532$ inch.
$M = P.e = 10 \times {\cdot}532 = 5{\cdot}32$ ton-inches.
$y_c = 3{\cdot}032''$; $y_t = {\cdot}968''$.

$$\therefore \quad p_c = \frac{M.y_c}{I_{xx}} = \frac{5{\cdot}32 \times 3{\cdot}032}{6{\cdot}07} = 2{\cdot}657 \text{ tons/in}^2\text{., compressive.}$$

$$p_t = \frac{M.y_t}{I_{xx}} = \frac{5{\cdot}32 \times {\cdot}968}{6{\cdot}07} = {\cdot}8484 \text{ ton/in}^2\text{., tensile.}$$

$$p_o = \frac{P}{A} = \frac{10}{4{\cdot}771} = 2{\cdot}096 \text{ tons/in}^2\text{., compressive.}$$

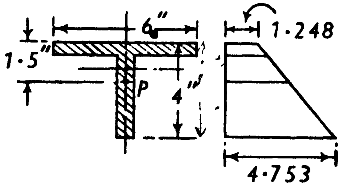


FIG. 154

$p_{max} = p_o + p_c = 2{\cdot}096 + 2{\cdot}657 = 4{\cdot}753$ tons/in^2., compres.
$p_{min} = p_o - p_t = 2{\cdot}096 - {\cdot}8484 = 1{\cdot}248$ tons/in^2., compres.
The stress distribution is shown in fig. 154.

Example 4.

A flat bar 5″ wide and ½″ thick carries an axial pull of 12 tons. A ½″ diameter hole is punched with its centre at a distance of 1½″ from the axis of the bar. Find the maximum and minimum stress intensities at the weakest section.

At the weakest section, area of section

$A = (5 - \frac{1}{2}) \times \frac{1}{2} = 2{\cdot}25 \text{ in}^2.$

$\therefore \quad p_o = \frac{P}{A} = \frac{12}{2{\cdot}25} = 5{\cdot}33 \text{ tons/in}^2\text{., tensile.}$

Let $\bar{x}$ be the distance of G the centre of gravity of the net section from the axis YY of the bar.

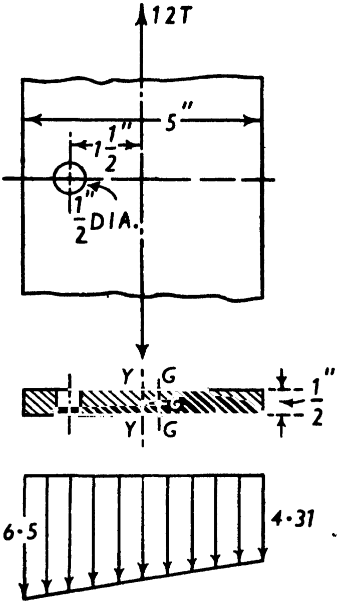


FIG. 155

The net section consists of:

(i) area $a_1 = 5 \times \frac{1}{2} = 2{\cdot}5$ in²., distance of its *c.g.* from the axis being zero,

less (ii) area $a_2 = \frac{1}{2} \times \frac{1}{2} = {\cdot}25$ in²., x_2 the distance of its *c.g.* from the axis YY being $-\frac{3}{2}''$.

$$\therefore \quad \bar{x} = \frac{a_1 x_1 - a_2 x_2}{a_1 - a_2} = \frac{0 - \frac{1}{4}\,(-\frac{3}{2})}{2{\cdot}5 - {\cdot}25} = \frac{1}{6}''.$$

The eccentricity of the load line from the *c.g.* of the net section, is, thus, $\frac{1}{6}''$.

$M = P.e = 12 \times \frac{1}{6} = 2$ ton-inches.

$y_c = 2{\cdot}5 - \frac{1}{6} = 2{\cdot}33''$

$$y_t = 2{\cdot}5 + \tfrac{1}{6} = 2{\cdot}67''$$

$$I_{yy} = \tfrac{1}{12} \times \tfrac{1}{2} \times 5^3 - \{\tfrac{1}{12} \times \tfrac{1}{2} \times (\tfrac{1}{2})^3 + \tfrac{1}{2} \times \tfrac{1}{2} \times (1{\cdot}5)^2\}$$

$$= \tfrac{125}{24} - \{\tfrac{1}{192} + \tfrac{9}{16}\} = \tfrac{125}{24} - \tfrac{109}{192} = 4{\cdot}64 \text{ in}^4.$$

$$\therefore\ I_{GG} = I_{yy} - 2{\cdot}25\ (\tfrac{1}{6})^2 = 4{\cdot}64 - {\cdot}0625 = 4{\cdot}58 \text{ in}^4.$$

$$p_c = \frac{My_c}{I_{GG}} = \frac{2 \times 2{\cdot}33}{4{\cdot}58} = 1{\cdot}017 \text{ tons/in}^2\text{., compressive.}$$

$$p_t = \frac{M.y_t}{I_{GG}} = \frac{2 \times 2{\cdot}67}{4{\cdot}58} = 1{\cdot}166 \text{ tons/in}^2\text{., tensile.}$$

$$p_{max} = p_o + p_t = 5{\cdot}33 + 1{\cdot}166 = 6{\cdot}5 \text{ tons/in}^2\text{., tensile.}$$

$$p_{min} = p_o - p_c = 5{\cdot}33 - 1{\cdot}017 = 4{\cdot}31 \text{ tons/in}^2\text{., tensile.}$$

The stress distribution is shown in fig. 155.

4 Wind pressure on walls and chimney shafts: The intensity of stress set up on the base of a wall or chimney shaft to resist its own weight is $p_o = \frac{W}{A}$, where W is the weight and A the area of the base section. If the density of masonry is ρ, $W = \rho.A.h.$, where h is the height of wall or shaft of uniform section A. The intensity of direct stress p_o is, therefore, equal to ρh. If there is a horizontal wind pressure of uniform intensity p acting on a side of width b, the force of wind $P = p.b.h$. This will exert a bending moment on the base of $P.\frac{h}{2}$. The base section has therefore to resist the direct load W accompanied by a moment $M = \frac{P.h}{2}$. The stress intensities at the base will, therefore, be

$$p_{max} = p_o + p_b$$

$$p_{min} = p_o - p_b, \text{ where } p_b = \frac{M}{Z},$$

Z being the modulus of the base section.

Example 5.

A masonry wall, 20 feet high, is of solid rectangular section, 12 feet wide and 4 feet thick. A horizontal wind pressure of 30 lb. per sq. foot acts on the 12 ft. side. Find the maximum and the mini-

mum stress intensities induced on the base, if the density of masonry is 140 lb. per cubic foot.

$$p_o = \rho h = 140 \times 20 = 2800 \text{ lb-ft}^2.,$$

$$P = p.b.h = 30 \times 12 \times 20 = 7200 \text{ lb-wt.}$$

$$M = \frac{P.h}{2} = 7200 \times 10 = 72000 \text{ lb-feet.}$$

$$Z = \tfrac{1}{6}bd^2 = \tfrac{1}{6} \times 12 \times 4^2 = 32 \text{ ft}^3.$$

$$\therefore \quad p_b = \frac{M}{Z} = \frac{72000}{32} = 2250 \text{ lb/ft}^2.$$

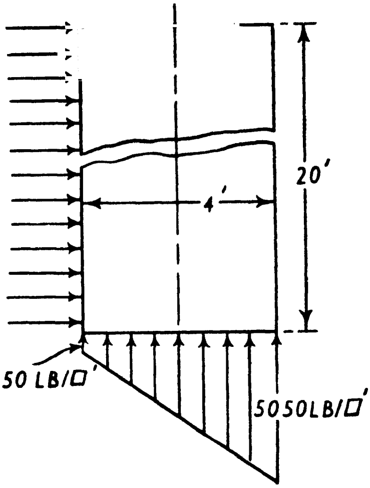


FIG. 156

$$\therefore \quad p_{max} = p_o + p_b = 2800 + 2250$$
$$= 5050 \text{ lb/ft}^2., \text{ compressive.}$$
$$p_{min} = p_o - p_b = 2800 - 2250$$
$$= 550 \text{ lb-ft}^2., \text{ compressive.}$$

The stress distribution is shown in fig. 156.

Example 6.

A square chimney 80 feet high, has an opening of 4 ft. by 4 ft. inside. Find the necessary thickness of brickwork at the base if the maximum permissible stress on brick masonry is limited to 7 tons/ft^2. The horizontal intensity of wind pressure is 28 lb/ft^2. and density of masonry is 126 lb/ft^3.

Let the external dimension of the shaft be D feet. The intensity of direct stress $p_o = \rho h = \frac{126 \times 80}{2240} = 4{\cdot}5$ tons/ft².

Since p_{max} is limited to 7 tons/ft²., the maximum permissible value for p_b is $7 - 4{\cdot}5 = 2{\cdot}5$ tons/ft². The force of wind pressure on a side of the chimney shaft is given by,

$$P = p.D.h = 28 \times 80 \times D = 2240D \text{ lb-wt.}$$
$$= D \text{ ton-wt.}$$
$$M = \frac{P.h}{2} = \frac{D \times 80}{2} = 40D \text{ ton-feet.}$$
$$Z = \frac{D^4 - d^4}{6D} = \frac{D^4 - 256}{6D} \text{ ft}^3.$$
$$\therefore \quad p_b = \frac{M}{Z} = \frac{40D \times 6D}{D^4 - 256} = \frac{240D^2}{D^4 - 256}$$

Equating, $\frac{240D^2}{D^4 - 256} = 2{\cdot}5$

$$\therefore \quad D^4 - 256 = 96D^2$$
$$\therefore \quad (D^2 - 48)^2 = 256 + 2304$$
$$= 2560$$
$$\therefore \quad D^2 - 48 = 50{\cdot}59$$
$$\therefore \quad D^2 = 98{\cdot}59$$

and $D = 9{\cdot}928$ feet, say, 10 feet.

The thickness of brickwork at the base

$$= \frac{D - d}{2} = \frac{10 - 4}{2} = 3 \text{ feet.}$$

5. Coefficient of wind-resistance: Wind pressure is usually specified as a horizontal force on a unit area of a vertical plane, on which it acts normally. The shape of the object exposed to the wind will however, affect the magnitude of the force exerted by the wind on it. If the area presented to the wind pressure is curved, as in the case of a cylindrical chimney shaft, the magnitude of the force will be less than when the area is a flat surface. The reduction factor c, depending upon the shape of the area exposed to the wind, is called the coefficient of wind-resistance. Its value varies from ·5 to ·75. For cylindrical shafts, *it is usually taken as ·6, though theoretically its* value will be $\frac{2}{3}$. This can be proved as follows:

Let p be the conventional intensity of horizontal wind pressure per square foot of a plane vertical area on which it acts normally. Let d be the external diameter and h the height of a cylindrical shaft.

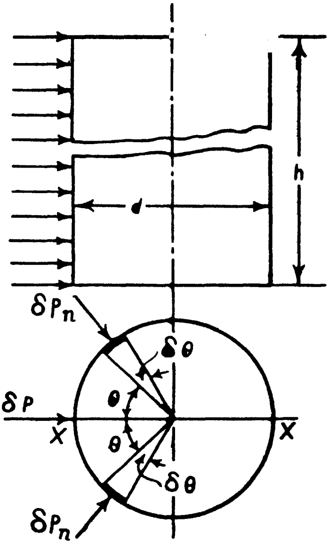


FIG. 157

Consider elementary strips of the exposed surface, subtending an angle $\delta\theta$ at the centre, at angle θ on either side of the axis XX (fig. 157).

The force of wind reaching each strip is $p.r\delta\theta.\cos\theta.h$, where r is the external radius. Resolving normally and tangentially, the normal component δP_n on each strip $= (p.r\delta\theta.\cos\theta.h)\cos\theta$, the tangential component δP_t blowing off. Compounding the two normal components δP_n into a single resultant δP, we have,

$$\delta P = 2\delta P_n.\cos\theta \text{ in the direction of } XX.$$

$$\therefore\quad \delta P = 2p.r.h.\cos^3\theta.\delta\theta.$$

Integrating over the whole exposed surface from $\theta = 0$ to $\theta = 90°$, we have,

$$P = \Sigma \delta P = \int_{0}^{\frac{\pi}{2}} 2.p.r.h.\ \cos^3\theta d\theta$$

$= p.d.h.\frac{2}{3}$, which proves the proposition.

Example 7.

A masonry chimney, 60 feet high, of uniform circular section, 12 feet external diameter, 5 feet internal diameter has to withstand a horizontal wind pressure of 24 lb. per square foot of projected area. Find the maximum and minimum stress intensities at the base, if the density of masonry is 130 lb. per cubic foot.

$p_o = \rho h = 130 \times 60 = 7800$ lb-ft².

$P = 24 \times 12 \times 60 = 17280$ lb-wt.

$$M = \frac{P.h}{2} = 17280 \times \tfrac{60}{2} = 518400 \text{ lb-feet.}$$

$$Z = \frac{\pi(D^4 - d^4)}{32D} = \frac{\pi(144+25)\ (144-25)}{32 \times 12} = 164\cdot6 \text{ lb-ft}^3.$$

$$\therefore \quad p_b = \frac{M}{Z} = \frac{518400}{164\cdot6} = 3150 \text{ lb/ft}^2.$$

$p_{max} = p_o + p_b = 7800 + 3150 = 10950$ lb/ft²., compressive.

$p_{min} = p_o - p_b = 7800 - 3150 = 4650$ lb/ft²., compressive.

Example 8.

A cylindrical steel chimney, 120 feet high, 5 feet external diameter, is exposed to a horizontal wind pressure whose intensity varies as the square root of the height above the ground. At a height of 49 ft. the intensity of wind pressure on a flat surface is 42 lb. per square foot. If the coefficient of wind-resistance is ·6, calculate the bending moment at the foot of the chimney.

Consider an element of the exposed surface of thickness δx at a height x above the ground. The intensity of wind pressure $p = k\sqrt{x}$, where k is a constant. To evaluate k, $p = 42$ lb/ft²., at $x = 49$ feet.

$$\therefore \quad 42 = k\sqrt{49}$$

$$\therefore \quad k = 6 \text{ and } p = 6\sqrt{x}.$$

On the element the force of wind pressure,

$$\begin{aligned}\delta P &= c.p.d.\delta x\\ &= \cdot 6 \times 6\sqrt{x} \times 5 \times \delta x\\ &= 18\sqrt{x}\ dx.\end{aligned}$$

The moment of δP about the base is $x\delta P$ or $18x^{\frac{3}{2}}\ dx$.

Summing up the moments of all these elementary forces δP, we get the total moment,

$$M = \Sigma x.\delta P = \int_0^{120} 18x^{\frac{3}{2}} dx$$

$$= 18\left[\tfrac{2}{5}x^{\frac{5}{2}}\right]_0^{120} = \tfrac{36}{5}\,[120 \times 120 \times \sqrt{120}] = 1135000 \text{ lb-ft.}$$

Example 9.

A chimney shaft 100 feet high, tapers from 10 feet external diameter at the base to 4 feet external diameter at the top. If the horizontal intensity of wind pressure is 50 lb. per square foot of flat surface, calculate the bending moment at the base due to the force of the wind, taking the coefficient of wind-resistance as $\frac{2}{3}$.

Consider an element of the exposed surface of thickness δx at a height x feet above the ground. The external diameter at this height is $(10 - \frac{6x}{100})$ feet. The force of wind pressure on the elementary strip is,

$$\begin{aligned}\delta P &= c.p\ (10 - \frac{6x}{100})\ \delta x\\ &= \tfrac{2}{3} \times 50\ (10 - \frac{6x}{100})\ \delta x.\end{aligned}$$

The bending moment at the base,

$$M = \Sigma x.\delta P = \int_0^{100} \frac{100x}{3}\ (10 - \frac{6x}{100})\ dx$$

$$= \frac{100}{3}\left[\ 5x^2 - \frac{2x^3}{100}\right]_0^{100} = \frac{100^3}{3}\,[5 - 2]$$

$$= 1000000 \text{ lb-feet.}$$

6. Water and earth pressure: Walls retaining water or earth etc. have lateral forces exerted on them due to the pressure of the retained material. The base of such a wall will have, besides the dead load coming on it, to resist the bending moment due to lateral pressure. Stability of dams and retaining walls will be studied, in detail, later. It is sufficient to indicate here that the problem is one of direct and bending stresses.

Example 10.

A masonry dam, 20 feet high, 4 feet wide at the top and 12 feet wide at the base has its water-face vertical and retains water to a depth of 18 feet. Calculate the maximum and minimum stress intensities induced at the base if the density of masonry is 140 lb/ft^3. and that of water 62·5 lb/ft^3.

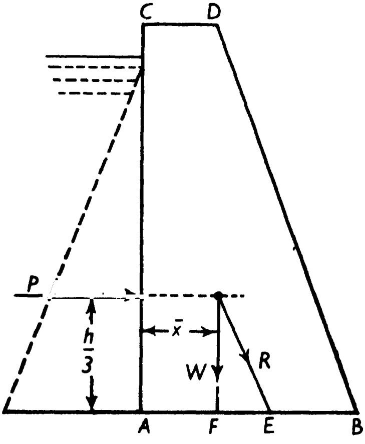


FIG. 158

Consider a foot-length of the wall. The weight of masonry

$W = 140 \times \frac{1}{2} \times (12 + 4) \times 20$

$= 22400$ lb-wt.

To locate its line of action, divide the dam section into,

(i) a rectangle 4′ × 20′, the distance of its c.g. from the face AC being 2 feet, and

(ii) a triangle whose area is $\frac{1}{2}(12-4) \times 20$ i.e. 80 sq. feet with its c.g. at $(4+\frac{1}{3} \times 8)$ feet from *AC*. The distance of the c.g. of the section from *AC* is given by,

$$\bar{x} = \frac{a_1x_1 + a_2x_2}{a_1 + a_2} = \frac{80 \times 2 + 80 \times \frac{20}{3}}{80 + 80} = \tfrac{13}{3} = 4{\cdot}33 \text{ feet.}$$

On the water face, the intensity of water pressure is zero at the surface increasing uniformly to wh at the bottom, where h is the depth of water impounded and w the density of water. Per foot length of wall, the lateral thrust P due to water pressure = average intensity of pressure × the area $= \frac{1}{2}wh.h = \frac{wh^2}{2}$ lb-wt. acting at $\frac{h}{3}$ from the bottom.

In our case,

$$P = \frac{62{\cdot}5 \times 18 \times 18}{2} = 10125 \text{ lb-wt. at 6 feet from } A.$$

The forces W and P can be compounded into a single resultant R. Let E be the point of incidence of R on the base AB and let its distance from A be z. The force R will have to be balanced by an equal and opposite reaction at the base AB. Resolving the resultant R normally and tangentially at E, the normal component W will be balanced by the normal stress intensities at the base, while the tangential component P will be balanced by the frictional resistance against sliding.

The effect of the water thrust P has thus been to shift the incidence of the load line from F to E. The eccentricity e of the load on the base is $(z-\frac{b}{2})$ where $AE = z$ and b is the bottom width of the section. To locate E, take moments about A of W and P and equate the sum to the moment of the resultant.

We have,

$$W.z = W\bar{x} + \frac{h}{3} \times P$$

or

$$z = \bar{x} + \frac{h}{3} \times \frac{P}{W}$$

In our problem,

$$z = 4{\cdot}33 + \frac{6 \times 10125}{22400} = 4{\cdot}33 + 2{\cdot}71 = 7{\cdot}04 \text{ feet.}$$

$$\therefore \quad e = 7{\cdot}04 - \frac{12}{2} = 1{\cdot}04 \text{ feet.}$$

On the base,

$$p_o = \frac{W}{A} = \frac{W}{b} = \frac{22400}{12} = 1867 \text{ lb/ft}^2.$$

$$M = W.e = 22400 \times 1{\cdot}04 \text{ lb-feet.}$$

$$Z = \tfrac{1}{6}db^2 = \tfrac{1}{6} \times 1 \times 12 \times 12 = 24 \text{ ft}^3.$$

$$\therefore \quad p_b = \frac{M}{Z} = \frac{22400 \times 1{\cdot}04}{24} = 970{\cdot}7 \text{ lb-ft}^2.$$

$\therefore$ $p_{max} = p_o + p_b = 1867 + 970{\cdot}7 = 2837{\cdot}7$ lb/ft^2., compressive at *B*.

$p_{min} = p_o - p_b = 1867 - 970{\cdot}7 = 896{\cdot}3$ lb/ft^2., compressive at *A*.

EXAMPLES VIII

(1) The section of a masonry pier 16 feet high is a hollow rectangle, external dimensions 12′ × 5′, internal dimensions 10′ × 3′. A horizontal thrust of 2½ tons is exerted at the top of the pier in the vertical plane bisecting the length. If the density of masonry is 140 lb. per cubic foot, calculate the maximum and minimum stress intensities at the base.

(1·976, ·0243 ton/ft^2., compressive.)

(2) A 22″ × 7″ × 75 lb. R.S.J. is used for the vertical post of a crane. When the crane is lifting a load of 5 tons, the distance of the load line from the c.g. of the section being 12 feet, find the extreme intensities of stress induced in the section, given that, for the 22″ × 7″ R.S.J., area of the section $A = 22{\cdot}06$ in^2. and $I = 1676{\cdot}8$ in^4.

(4·95 tons/in^2., compressive; 4·5 tons/in^2., tensile.)

(3) A hollow C.I. column of rectangular section, 24″ × 12″ overall and 21″ × 10″ internally, carries a load of 150 tons which is off the geometric axis by 4″ in the vertical plane bisecting the thickness. Calculate the extreme intensities of stress induced in the section.

(3·1; ·743 tons/in^2., compressive.)

(4) A short wooden pillar is rectangular in section, 15″ wide and 9″ thick. It carries at the top two point-loads P_1 and P_2 in the vertical plane bisecting the thickness, P_1 acting at 2″ from the centre on one side and P_2 at 3″ on the other side. If the stress is throughout compressive, and the extreme intensity of stress on the side in which P_1 acts is three times the extreme intensity on the other side, calculate the ratio of P_1 and P_2. (17:3.)

✓(5) A short hollow C.I. column, 9″ external diameter, thickness of metal 1″, carries an axial load of 25 tons. It also carries a load of 5 tons on a bracket, the load line 15″ from the axis of the column. Find the maximum and minimum stress intensities on the section.
(2·85 tons/in^2., compressive; ·46 ton/in^2., tensile.)

(6) The principal section of a crane-hook is a trapezium of depth 3″, inner width 2½″ and outer width 1½″. If the distance of the load line is 2″ from the intrados, calculate the safe load in tons if the maximum permissible tensile stress is 6 tons/in^2. What then is the minimum stress induced in the section? Neglect the effect of initial curvature of the hook. (4·92 tons; 5·29 tons/in^2.)

(7) A short hollow cylindrical C.I. column is 12″ external and 9″ internal diameter. In casting, the bore got eccentric so that the thickness varies from 1″ at one end to 2″ at the other. If the column carries a load of 80 tons along the axis of the bore, calculate the extreme intensities of stress induced in the section.
(2·54 ton/in^2; 0·877 ton/in^2., compressive.)

(8) A 12″ × 8″ × 65 lb. R.S. section is used for a column to transmit a load of 60 tons. The load line is eccentric, being 2″ above *XX* and 1″ to the right of *YY*. Find the maximum and minimum stresses induced in the section, given area of section $= 19{\cdot}12$ in^2.; $I_{xx} = 487{\cdot}80$ in^4.; $I_{yy} = 65{\cdot}20$ in^4.
(8·3 tons/in^2., compressive; 2·018 tons/in^2., tensile.)

(9) A tie bar 9″ wide and ¾″ thick transmits an axial pull of 30 tons. A hole 1½″ diameter is drilled through the bar with its centre at a distance of 2″ from the axis of the bar. Find the maximum stress induced in the material. (6·8 tons/in^2., tensile.)

(10) A masonry pillar *d* feet in diameter is subjected to a horizontal intensity of wind pressure of *p* lb/ft^2. If the coefficient of wind-resistance is *c*, prove that the maximum permissible height for the pillar so that no tension is induced at the base, is given in feet by $h = \dfrac{\rho.\pi.d^2}{16c.p}$, where ρ is the density of masonry.

(11) A cylindrical chimney shaft of a hollow circular section, 7 feet external diameter, 3 feet internal diameter, is 80 feet high. If

the horizontal intensity of wind pressure varies as $x^{\frac{2}{7}}$ where x is the vertical height above the ground, calculate the overturning moment at the base due to the force of wind pressure, taking the coefficient of wind-resistance as ·6. Given that the horizontal intensity of wind pressure at a height of 64 feet is 20 lb/ft². If the density of masonry is 140 lb/ft³., calculate the extreme intensities of stress at the base.

(104·4 ton-ft; 8·21, 1·79 tons/ft²., compressive.)

(12) A tapering chimney of circular section 150 feet high, 12 feet external diameter at the base and 8 feet external diameter at the top is subjected to a uniform wind pressure of 21 lb. per square foot of projected area. Calculate the overturning moment at the base. If the weight of the chimney is 600 tons and the internal diameter at the base is 4 feet, calculate the maximum and minimum stress-intensities induced in the base section.

(984·4 ton-ft; 11·84; ·09 ton/ft²., compressive.)

(13) A masonry wall, 8 feet long, trapezoidal in section with one side vertical, is 20 feet high, 4 feet wide at the top and 12 feet at the bottom. A thrust of 17·32 tons is transmitted at the top, on the vertical side, in the middle of the length, at an angle of 30° to the horizontal. If masonry weighs 140 lb. per cubic foot, calculate the extreme stress-intensities on the base.

(1·524, ·3232 ton/ft²., compressive.)

(14) A masonry retaining wall, 21 feet high, is trapezoidal in section, 3 feet wide at the top and 8 feet at the base, with one side vertical. If the lateral pressure exerted by the retained material on the vertical face varies from zero at the top to 500 lb. per square foot at the base, calculate the maximum and minimum intensities of stress induced in the base, the density of masonry being 130 lb. per cubic foot.

$\left\{\begin{array}{l}\text{3828 lb/ft}^2\text{., compressive.}\\ \text{75 lb/ft}^2\text{., tensile.}\end{array}\right.$

(15) A steel rod 1″ diameter is placed inside a cast iron tube 3″ outside diameter and 2″ inside diameter, 8 feet long. The rod passes through a rigid cover plate at each end of the tube, and nuts are screwed on to the projecting ends of the rod so that the cover plates bear on the ends of the tube. Find the maximum stress in the cast iron tube, if one of the nuts is tightened so as to stretch the rod $\frac{1}{8}$″,

(a) if the rod is concentric with the tube

(b) if it is $\frac{1}{4}$″ out of centre.

E for C.I. $= 15 \times 10^6$ lb/in².

E for steel $= 30 \times 10^6$ lb/in². (L.U.)

{(a) 7812 lb/in². (b) 11420 lb/in².}

(16) A 13″ × 5″ B.S.B. when used as a short column can carry safely an axial load of 50 tons. If the column is strengthened by riveting a 10″ × $\frac{1}{2}$″ steel plate to each flange in order that the load of 50 tons may be permitted to act in a direction parallel with the axis of the column through a point in the centre line of the web, find the maximum eccentricity possible if the compressive stress in the compound section is not to exceed that when the axial load acts on the unplated section. For a 13″ × 5″ B.S.B., $A = 10{\cdot}30$ in²; $I_{xx} = 283{\cdot}5$ in⁴. (L.U.)
(5·05″.)

(17) A bar of *T*-section as shown in fig. 158(*a*) is subjected to a longitudinal pull *P* applied at a point on the *yy*-axis but not on the centroid of the section. Determine the magnitude of *P* and the position of its line of action if the stresses across the section vary from 0·620 ton/in². compression at the top to 7·450 tons/in². tension at the bottom. (L.U.)
(*P* = 3·78 tons; 2·126″ from top.)

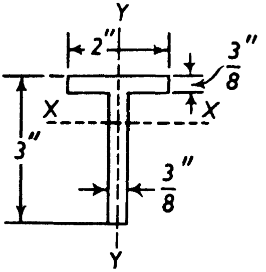


FIG. 158(*a*)

(18) Fig. 158(*b*) shows a vertical steel mast *AB* of tubular section, 6$\frac{1}{2}$″ external diameter and 6″ internal diameter, fixed at its

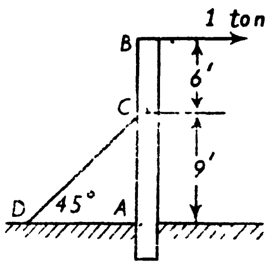


FIG. 158(*b*)

base. A horizontal force of 1 ton acts at the top and the mast is stayed by a guy rope *CD* in the same vertical plane as the horizontal force. Calculate the pull in the guy rope and the maximum stress in the mast if the top end B is deflected 2″ in the direction of the 1–ton load. $E = 13000$ tons/in². (L.U.)

(2·22 tons; 10·06 tons/in².)

(19) A brick chimney weighs 160 tons and has internal and external diameters at the base of 6 feet and 10 feet respectively. Owing to settlement, the chimney leans to one side and the centre of gravity is 6″ from the vertical through the centre of the base. Calculate the maximum horizontal force applied 40 feet above the base in order that there shall be no tension in the chimney. (L.U.)

(4·8 tons.)

(20) A tie rod of constant circular section is required to withstand a maximum tension of 50·3 tons but the end fixing is such that the line of action must be offset 0·3″ from the axis. Determine to the nearest $\frac{1}{8}$″, the minimum diameter of the tie rod if the maximum allowable stress is 8 tons/in². (L.U.)

{ $3\frac{5}{8}$″. }

(21) A 12″ × 6″ B.S.B. is used as a stanchion to carry two loads W and $2W$, which act in directions parallel with the axis of the stanchion. The load W acts on the *yy*-axis and 8″ from the *xx*-axis.

(a) Find the boundary within which $2W$ must act if there is to be no tension.

(b) If $2W$ acts on the *xx*-axis, find the magnitude of W and the position of the line of action of $2W$ if the greatest and least stresses are 7 tons/in². and $\frac{1}{2}$ ton/in²., both compressive. $I_{xx} = 375\cdot8$ in⁴.; $I_{yy} = 28\cdot28$ in⁴.; sectional area $= 15\cdot89$ in². Make a diagram of the section showing the load positions and the boundary required. (L.U.)

$$\left\{\begin{array}{l}\text{(a) Boundary lines bounded by } \left\{\begin{array}{l}0,1\cdot914'';\ 0,-9\cdot914''; \\ \cdot288'',\ 0;-\cdot288'',\ 0.\end{array}\right\} \\ \text{(b) } W = 19\cdot68 \text{ tons};\ \ (\pm\cdot17'', 0.)\end{array}\right\}$$

(22) A 15″ × 6″ B.S.B. is used as a column to carry a load acting in a direction parallel with the longitudinal axis of the column but not passing through either of the principal axes of the section. The magnitude and line of action of the load are such that the stresses at the corners A, B and C are as shown in fig. 158(*c*). Determine the magnitude and location of the load, the stress at D and the posi-

tion of the neutral axis and make a diagram of the section showing these values and the reference letters. For a 15″ × 6″ B.S.B., $I_{xx} = 491{\cdot}9$ in^4.; $I_{yy} = 19{\cdot}87$ in^4.; $A = 13{\cdot}24$ in^2. (L U.)

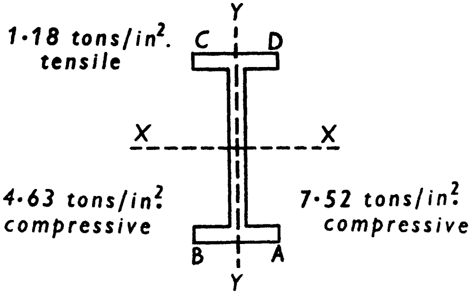


FIG. 158(*c*)

$P = 42$ tons at $x = +\cdot 228''$, $y = -4\,537''$;
Stress at $D = 2{\cdot}71$ tons/in^2., compressive;
$\cdot 482\,x - \cdot 387y + 3{\cdot}172 = 0$.

CHAPTER IX

COLUMNS AND STRUTS OF UNIFORM SECTION

1. Axial loading: For a short column under axial loading the relation between the load P and the stress induced is given by $P = p.A$, where A is the area of resisting section and p the intensity of stress. If the column is loaded to destruction, the crushing load $P_c = f_c.A$, where f_c is the ultimate crushing stress. Adopting a suitable factor of safety, a safe working stress for the material may be selected. For ordinary struts and columns, however, it cannot be said that failure will occur purely by crushing. At a certain critical load, an ordinary strut will start bending. Once this process of buckling commences, failure is inevitable. This critical or bucking load for a given section and length of strut is less than the crushing load P_c for failure by pure compression. Its value depends on how the ends are gripped and upon the ratio of its length and the dimensions of its section.

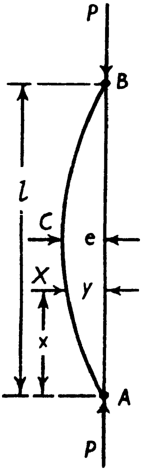


FIG. 159

It is therefore necessary to investigate this crippling load for a strut of given length and section for given end-conditions.

Consider a straight column of length l and sectional area A. Let it be loaded axially by a load which increases gradually until at some load P, elastic instability occurs and the column bends. Referring to fig. 159, at a section X, at a distance of x from A, the load P is now eccentric by y. The section has therefore to resist an axial load P accompanied by a moment $P.y$. The section which has the largest moment coming on it, is at C the middle of the column where the eccentricity is e. The maximum and minimum stress intensities will be given by,

$$p_{max} = p_o + p_b = \frac{P}{A} + \frac{P.e}{Z}$$

and
$$p_{min} = p_o - p_b = \frac{P}{A} - \frac{P.e}{Z}$$

Failure will occur either by p_{max} i.e. $(p_o + p_b)$ reaching the ultimate crushing stress f_c for the material or $(p_o - p_b)$ becoming tensile and reaching the ultimate tensile stress for the material. In either case, failure is not due merely to p_o or the direct stress offered to the critical load but also to the added effect of the bending moment accompanying the buckling of the column.

2. Very long columns — Euler's formulae: We shall first consider the case of very long columns for which the crippling load is low, so that the effect of direct stress p_o is negligible as compared to p_b the bending stress. In other words, we shall consider the failure to occur because of bending only.

We shall consider a straight column of length l. The crippling load P at which it starts bending will depend upon the sectional area and the end conditions. We shall consider four standard cases.

Case I. One end fixed, the other free:

Let the end A be direction-fixed as well as position-fixed while the end B is free. Let P be the critical load under which the column deflects and let a be the deflection of the free end B from the original centre-line of the column. Referring to fig. 160(*a*), consider a section *X* of the column at a distance x from the fixed end *A* and let y be its deflection from the initial centre-line. The load P is now acting on the section at an eccentricity of $(a - y)$. The bending moment at the section is $P(a - y)$ and we have,

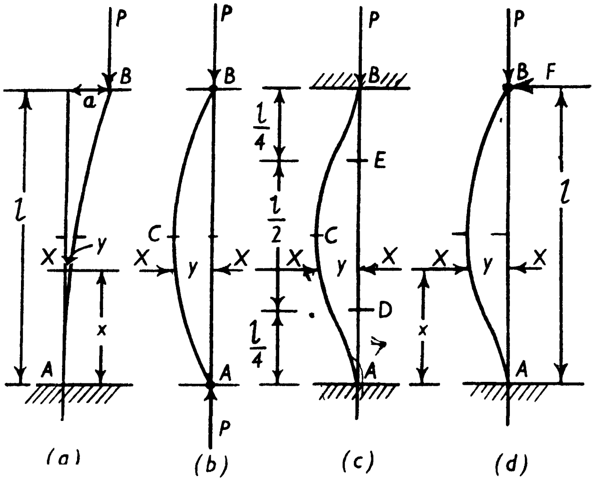


FIG. 160

$EI\frac{d^2y}{dx^2} = + P(a - y)$, according to the convention we have adopted.

Re-arranging,

$$\frac{d^2y}{dx^2} + \frac{P}{EI}.y = \frac{P.a}{EI}.$$

Solving the differential equation,

$y = C_1\cos\sqrt{\frac{P}{EI}}x + C_2\sin\sqrt{\frac{P}{EI}}x + a$, where C_1 and C_2 are constants of integration. To evaluate the constants we have the end conditions at *A*, where $x = 0$ and $y = 0$ and also

$$\frac{dy}{dx} = 0.$$

$$\therefore \quad 0 = C_1 + a$$

$$\therefore \quad C_1 = - a.$$

Also, $\frac{dy}{dx} = -C_1 . \sqrt{\frac{P}{EI}} \sin \sqrt{\frac{P}{EI}}\, x + C_2 \sqrt{\frac{P}{EI}} \cos \sqrt{\frac{P}{EI}} x$

$\therefore \quad 0 = C_2 \sqrt{\frac{P}{EI}}$

$\therefore \quad C_2 = 0$, since P is not zero.

The column therefore takes up a cosine-curve given by

$$y = a\left(1 - \cos x\sqrt{\frac{P}{EI}}\right)$$

At B, where $x = l$, $y = a$

$$\therefore \quad a = a\left(1 - \cos l\sqrt{\frac{P}{EI}}\right)$$

$$\therefore \quad \cos l\sqrt{\frac{P}{EI}} = 0$$

i.e. $\quad l\sqrt{\frac{P}{EI}} = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots\ldots\ldots$

Taking the first value, the least magnitude of P is given by,

$$l\sqrt{\frac{P}{EI}} = \frac{\pi}{2}$$

or $\quad P = \frac{\pi^2 EI}{4l^2}$, which is the crippling load.

Case II. Both ends free or hinged:

If the ends A and B are free and axial loading commences, they will no longer be position-free but direction-free and, at the critical load P, the column will bend as if the ends were hinged — not free to move out of position but free to adopt another angular position.

Consider a section X at a distance x from A and let y be its deflection from the centre-line as in fig. 160(*b*). The eccentricity of the load P on the section is now y and therefore the bending moment is $P.y$. Since the column has bent concave to its original centre-line, according to the convention we have adopted,

$$EI\frac{d^2y}{dx^2} = -P.y$$

$$\therefore \quad \frac{d^2y}{dx^2} + \frac{P}{EI}.y = 0.$$

The solution of this differential equation is,

$$y = C_1 \cos x\sqrt{\frac{P}{EI}} + C_2 \sin x\sqrt{\frac{P}{EI}}.$$

At A, $x = 0$ and $y = 0$

$$\therefore \quad C_1 = 0$$

At B, $x = l$ and $y = 0$

$$\therefore \quad 0 = C_2 \sin l\sqrt{\frac{P}{EI}}$$

This is possible if C_2 is zero, in which case, the column has not bent at all or $\sin l\sqrt{\frac{P}{EI}} = 0$

$$\therefore \quad l\sqrt{\frac{P}{EI}} = 0,\ \pi,\ 2\pi,\ 3\pi, \ldots\ldots\ldots\ldots$$

Taking the least significant value,

$$l\sqrt{\frac{P}{EI}} = \pi$$

or $\quad P = \frac{\pi^2 EI}{l^2}$, which is the crippling load in this case.

Case III. Both ends fixed:

Fig. 160(*c*) shows the deformation of the column. From *A* to *D*, for a length of $\frac{l}{4}$ it is convex; from *D* to *E* for a length of $\frac{l}{2}$ it is concave and for the remaining length of $\frac{l}{4}$ from *E* to *B*, it is convex towards the original centre-line *AB*. The column can therefore be looked upon as being equivalent either to (i) a column of length $\frac{l}{2}$ with hinged ends or to (ii) a column of length $\frac{l}{4}$ with one end fixed and the other free. The crippling load P will be

$$\frac{\pi^2 EI}{\left(\frac{l}{2}\right)^2} \text{ or } \frac{\pi^2 EI}{4\left(\frac{l}{4}\right)^2} \text{ i.e. } \frac{4\pi^2 EI}{l^2}.$$

We have derived the crippling load $\frac{4\pi^2 EI}{l^2}$ by looking upon the column as equivalent to cases I and II, the equivalent length being $\frac{l}{4}$ and $\frac{l}{2}$ respectively and quoting the formulae for crippling loads for these cases.

For a direct proof, consider, as usual, a section X at a distance of x from A and let the corresponding deflection be y. As the ends are fixed, there will be fixing couples M_A and M_B at the ends, as in the case of a built-in beam. These will be equal by symmetry. The bending moment at the section X will, therefore, be $M_A \quad P.y$.

We have, $EI\frac{d^2y}{dx^2} = M_A - P.y$

$$\frac{d^2y}{dx^2} + \frac{P}{EI}.y = \frac{M_A}{EI}.$$

Solving the equation,

$$y = C_1 \cos x\sqrt{\frac{P}{EI}} + C_2 \sin x\sqrt{\frac{P}{EI}} + \frac{M_A}{P}$$

$$\frac{dy}{dx} = -C_1\sqrt{\frac{P}{EI}} \sin x\sqrt{\frac{P}{EI}} + C_2\sqrt{\frac{P}{EI}} \cos x\sqrt{\frac{P}{EI}}$$

At A, $x = 0$, $y = 0$ and $\frac{dy}{dx} = 0$

$$0 = C_1 + \frac{M_A}{P}$$

and $$0 = C_2\sqrt{\frac{P}{EI}}$$

$\therefore$ $$C_2 = 0 \text{ and } C_1 = -\frac{M_A}{P}$$

The equation of the curve is, therefore

$$y = -C_1 \left\{1 - \cos x\sqrt{\frac{P}{EI}}\right\}$$

At B, $x = l$ and $y = 0$

$$\therefore \quad 0 = -C_1 \left\{1 - \cos l\sqrt{\frac{P}{EI}}\right\}$$

$\therefore$ Either C_1 is zero, in which case, the column has not bent at all, or $\left(1 - \cos l\sqrt{\frac{P}{EI}}\right)$ is zero, i.e,

$$\cos l\sqrt{\frac{P}{EI}} = 1$$

$$\therefore \quad l\sqrt{\frac{P}{EI}} = 0,\ 2\pi,\ 4\pi,\ 6\pi,\ \ldots \qquad \ldots(1)$$

At B also, $\frac{dy}{dx} = 0$

$$\therefore \quad 0 = -C_1\sqrt{\frac{P}{EI}} \sin l\sqrt{\frac{P}{EI}}$$

i.e. $\sin l\sqrt{\frac{P}{EI}} = 0$, since C_1 cannot be zero.

$$\therefore \quad l\sqrt{\frac{P}{EI}} = 0,\ \pi,\ 2\pi,\ 3\pi,\ \ldots \qquad \ldots(2)$$

The minimum value of $l\sqrt{\frac{P}{EI}}$ consistent both with (1) and (2) is 2π.

$$\therefore \quad l\sqrt{\frac{P}{EI}} = 2\pi$$

or

$$P = \frac{4\pi^2 EI}{l^2}.$$

We could also have obtained the result by putting $\frac{dy}{dx} = 0$ at C where $x = \frac{l}{2}$.

Thus,

$$0 = -C_1\sqrt{\frac{P}{EI}} \sin \frac{l}{2}\sqrt{\frac{P}{EI}}.$$

$$\therefore \quad \sin \frac{l}{2}\sqrt{\frac{P}{EI}} = 0, \text{ since } C_1 \text{ is not zero.}$$

$$\therefore \quad \frac{l}{2}\sqrt{\frac{P}{EI}} = 0,\ \pi,\ 2\pi, \ldots\ldots$$

Taking the minimum significant value,

$$\frac{l}{2}\sqrt{\frac{P}{EI}} = \pi$$

$$\text{or} \qquad P = \frac{4\pi^2 EI}{l^2}.$$

Case IV. One end fixed the other hinged:

Let the end A be fixed and B hinged as in fig. 160(d). Since B is position-fixed, though direction-free, it cannot retain its position unless the hinge exerts a horizontal force F on it. The bending moment at a section X at a distance x from A will be,

$$EI\frac{d^2y}{dx^2} = -P.y + F\ (l-x)$$

$$\therefore \quad \frac{d^2y}{dx^2} + \frac{P}{EI}.y = \frac{F\,(l-x)}{EI}.$$

Solving the equation,

$$y = C_1 \cos x\sqrt{\frac{P}{EI}} + C_2 \sin x\sqrt{\frac{P}{EI}} + \frac{F}{P}\ (l-x)$$

$$\therefore \quad \frac{dy}{dx} = -C_1\sqrt{\frac{P}{EI}}\sin x\sqrt{\frac{P}{EI}} + C_2\sqrt{\frac{P}{EI}}\cos x\sqrt{\frac{P}{EI}} - \frac{F}{P}$$

At A, $x = 0$, $y = 0$ and $\frac{dy}{dx} = 0$

$$\therefore \quad 0 = C_1 + \frac{Fl}{P}$$

$$\text{and} \qquad 0 = C_2\sqrt{\frac{P}{EI}} - \frac{F}{P}$$

$$\therefore \quad C_1 = -\frac{Fl}{P} \text{ and } C_2 = \frac{F}{P}\sqrt{\frac{EI}{P}}.$$

At B, $x = l$, $y = 0$

$$\therefore \quad 0 = C_1 \cos l\sqrt{\frac{P}{EI}} + C_2 \sin l\sqrt{\frac{P}{EI}}$$

Substituting for C_1 and C_2,

$$0 = -\frac{Fl}{P} \cos l\sqrt{\frac{P}{EI}} + \frac{F}{P}\sqrt{\frac{EI}{P}} \sin l\sqrt{\frac{P}{EI}}$$

$$\therefore \quad \tan l\sqrt{\frac{P}{EI}} = l\sqrt{\frac{P}{EI}}.$$

The value of $l\sqrt{\frac{P}{EI}}$ in radians has to be such that its tangent is equal to itself. The only angle whose tangent is equal to itself is about 4·5 radians.

$$\therefore \quad l\sqrt{\frac{P}{EI}} = 4{\cdot}5$$

or $\quad P = \frac{20\frac{1}{4}EI}{l^2}$, approximately $\frac{2\pi^2 EI}{l^2}$ which is the crippling load in this case.

Equivalent length:

We have worked out the crippling loads for columns of length l, under different end-conditions. It is enough to remember the value of $P = \frac{\pi^2 EI}{l^2}$ for the case of hinged ends. The others can be deduced from this standard case by using an equivalent length. Thus, a column of length l, one end fixed, the other free, is equivalent to a column with both ends hinged, the length being $2l$. Similarly, a column of length l with both ends fixed, is equivalent to one with both ends hinged, the corresponding length being $\frac{l}{2}$.

The equivalent lengths for the four cases already discussed may be stated as under:

	Case I	Case II	Case III	Case IV
Equivalent length:	$2l$	l	$\frac{l}{2}$	$\frac{l}{\sqrt{2}}$
Crippling load:	$\frac{\pi^2 EI}{(2l)^2}$ or $\frac{\pi^2 EI}{4l^2}$	$\frac{\pi^2 EI}{l^2}$	$\frac{\pi^2 EI}{(l/2)^2}$ or $\frac{4\pi^2 EI}{l^2}$	$\frac{\pi^2 EI}{(l\sqrt{2})^2}$ or $\frac{2\pi^2 EI}{l^2}$

These are the well-known Euler's formulae for long columns. As bending under the crippling load will take place about the axis of least resistance, the value of I in the formulae will be the least moment of Inertia of the section, unless special constraint compels the column to bend differently.

Taking the case of both ends hinged as standard, the crippling load $P = \frac{\pi^2 EI}{l^2} = \frac{\pi^2 E.A.k^2}{l^2}$. The crippling load per square inch of section is therefore given by $\frac{P}{A} = \frac{\pi^2 E}{\left(\frac{l}{k}\right)^2}$ which as will be seen, is not a constant but varies inversely as the square of $\frac{l}{k}$ where l is the length and k the minimum radius of gyration of the section. $\frac{l}{k}$ is known as the "Slenderness ratio" of the column.

These formulae have been theoretically derived on the assumption that the slenderness ratio $\frac{l}{k}$ is so large that the failure occurs due to bending, the effect of direct stress p_o being negligible. For mild steel, taking the crippling stress at 21 tons/in^2. and E as 13500 tons/in^2., we can deduce the value of the slenderness ratio upto and beyond which, Euler's formula is valid. For the case of hinged ends, the crippling load is $\frac{P}{A} = \frac{\pi^2 E}{\left(\frac{l}{k}\right)^2}$

$$\therefore \quad \frac{\pi^2 E}{\left(\frac{l}{k}\right)^2} \leqq 21$$

$$\text{or} \quad \left(\frac{l}{k}\right)^2 \geqq \frac{13500 \times \pi^2}{21}$$

$$\geqq 6346$$

$$\therefore \quad \frac{l}{k} \geqq 79{\cdot}66, \text{ say, } 80.$$

For both ends fixed,

$$\frac{P}{A} = \frac{4\pi^2 E}{\left(\frac{l}{k}\right)^2}$$

$$\therefore \quad \frac{4\pi^2 E}{\left(\frac{l}{k}\right)^2} \leqq 21 \quad \text{i.e.} \quad \frac{l}{k} \geqq 160.$$

Example 1.

A series of struts, both ends fixed, has slenderness ratio of 40, 60, 80....upto 200. Using Euler's formula, calculate the collapsing load per sq. inch of section, taking $E = 13000$ tons/in^2.

The crippling load per sq. inch is,

$$\frac{P}{A} = \frac{4\pi^2 E}{\left(\frac{l}{k}\right)^2} = \frac{513400}{\left(\frac{l}{k}\right)^2} \text{ tons.}$$

Substituting the given values of $\frac{l}{k}$, the crippling load in tons per sq. inch can be calculated and tabulated as under:

$\frac{l}{k}$:	40	60	80	100	120	140	160	180	200
$\frac{P}{A}$:	321	142·6	80·2	51·34	35·65	26·2	20·05	15·84	12·84

It will be seen from the table, that for a steel strut with both ends fixed, Euler's formula is valid for values of $\frac{l}{k}$ of about

160 and more. For smaller values of $\frac{l}{k}$ Euler's formula is not applicable as it gives too high a value for the crippling stress. The material will have already got crushed to destruction before reaching that load.

Example 2.

A mild steel tube 1″ internal diameter, thickness of metal $\frac{1}{8}$″, length 10 feet, is used as a strut, one end fixed, the other hinged. Calculate the collapsing load using $E = 13500$ tons/in².

$$I \text{ for the section} = \frac{\pi}{64}(D^4 - d^4) = \frac{\pi}{64}(1{\cdot}25^2 + 1)\ (1{\cdot}25^2 - 1)$$
$$= {\cdot}07076 \text{ in}^4.$$
$$P = 20\tfrac{1}{4}\frac{EI}{l^2} = \frac{20{\cdot}25 \times 13500 \times {\cdot}07076}{120^2}$$
$$= 1{\cdot}34 \text{ tons.}$$

3. Empirical formulae: Rankine's formulae: For a short column where failure will occur by pure compression, the ultimate load is given by $P_c = f_c.A$, where f_c is the crushing stress for the material. For a long column with such a slenderness ratio that the Euler formula is valid failure is due to bending and the crippling load is $P_e = \frac{\pi^2 EI}{l^2}$ for hinged ends. For ordinary struts and columns which can neither be classified as very short nor very long, failure will be due to combined direct and buckling stresses. To arrive at the crippling load in such cases, Rankine proposed an empirical relation which is named after him. The structure of the formula is so ingenious that it covers all cases ranging from very short to very long struts and columns, so that the crippling load by this formula approximates to P_c in the case of short struts and to P_e in the case of long ones.

For the crippling load P, Rankine proposed the relation, $\frac{1}{P} = \frac{1}{P_c} + \frac{1}{P_e}$, where $P_c = f_c.A$, the ultimate load for a short strut and P_e is the Eulerian crippling load being $\frac{\pi^2 EI}{l^2}$ for hinged ends.

In this expression, for a given column-section of a given material, P_c is a constant and therefore $\frac{1}{P_c}$ is constant. If the column is short, P_e being $\frac{\pi^2 EI}{l^2}$ is large and therefore its reciprocal $\frac{1}{P_e}$ is small. As l diminishes, $\frac{1}{P_e}$ also diminishes, until, in the expression $\frac{1}{P_c}+\frac{1}{P_e}$, $\frac{1}{P_e}$ becomes negligible as compared to $\frac{1}{P_c}$ which remains constant.

$\therefore$ As l diminishes,

$$\frac{1}{P} \rightarrow \frac{1}{P_c}$$

or $$P \rightarrow P_c.$$

On the other hand, if l is large, P_e is small and its reciprocal $\frac{1}{P_e}$ is large. As l increases, $\frac{1}{P_e}$ will also increase until in the expression $\frac{1}{P_c}+\frac{1}{P_e}$, $\frac{1}{P_c}$ which is constant becomes negligible as compared to $\frac{1}{P_e}$.

$\therefore$ As l increases,

$$\frac{1}{P} \rightarrow \frac{1}{P_e}$$

or $$P \rightarrow P_e$$

The value of P obtained from the formula $\frac{1}{P}=\frac{1}{P_c}+\frac{1}{P_e}$ will, therefore, cover all cases ranging from short to long columns and struts. The expression can be re-arranged as:

$$P=\frac{P_c\,P_e}{P_e+P_c}=\frac{P_c}{1+\dfrac{P_c}{P_e}}=\frac{f_c.A}{1+\dfrac{f_c.A.l^2}{\pi^2 E.I}}$$

$$= \frac{f_c.A}{1 + \frac{f_c}{\pi^2 E}\left(\frac{l}{k}\right)^2}.$$

For a given material, the expression $\frac{f_c}{\pi^2 E}$ is a constant. Replacing it by a, the formula becomes:

$P = \dfrac{f_c.A}{1 + a\left(\frac{l^2}{k^2}\right)}$, where f_c and a are constants for a given material. This is Rankine's classical formula for struts and columns.

It will be noted that in deriving the formula, we substituted $\frac{\pi^2 EI}{l^2}$ for P_e. The constant a in the formula is, therefore, for a column with both ends hinged. If the column has both ends fixed, the value of P_e is $\frac{4\pi^2 EI}{l^2}$ and the constant in the denominator of the Rankine formula will be $\frac{a}{4}$. Similarly, for a column with one end fixed and the other free, it will be $4a$; while for one end fixed and the other hinged, it will be $\frac{a}{2}$.

Though the formula is empirical, the value of the constant a is not derived from the values of f_c, π^2 and E but determined experimentally. The formula has therefore practical validity. The following table gives the values of Rankine's constants f_c and a for the materials most commonly used for struts and columns.

No.	Material	f_c tons/in^2.	a for ends hinged
1	Wrought iron	16	$\frac{1}{9000}$
2	Cast iron	36	$\frac{1}{1600}$
3	Mild steel	21	$\frac{1}{7500}$
4	Strong timber	3·2	$\frac{1}{750}$

If we examine the Rankine formula, it will be seen that the crippling load P is less than $f_c.A$ — the ultimate load for the column under pure compression with no possibility of buckling — the reduction-factor being $\dfrac{1}{1+a\dfrac{l^2}{k^2}}$. The value of the reduction-factor depends upon the slenderness ratio $\dfrac{l}{k}$. To arrive at the safe axial load for a strut of given length, section and end-conditions, use a suitable factor of safety for the material. The factor of safety is about 3 to 4 for mild steel.

Example 3.

A 10″ × 6″ × 40 lb. R.S.J. is used as a stanchion, length 15 feet, one end fixed, the other hinged. Using Rankine's constants and a factor of safety of 3, find the safe axial load for the stanchion. Area of section $= 11{\cdot}77$ *in*^{2}.; $I_{xx} = 204{\cdot}80$ *in*^{4}; $I_{yy} = 21{\cdot}76$ *in*^{4}.

The crippling load will be given by,

$$P = \frac{f_c.A}{1+\dfrac{a}{2}.\dfrac{l^2}{k^2}}, \text{ where } f_c = 21 \text{ tons/in}^2., \text{ and } a = \frac{1}{7500}.$$

To obtain k,

$$I_{yy} = Ak^2$$

$$\therefore\ 21{\cdot}76 = 11{\cdot}77k^2$$

$$\therefore\ k^2 = 1{\cdot}85 \text{ in}^2.$$

Substituting,

$$P = \frac{21 \times 11{\cdot}77}{1+\dfrac{1}{15000}\times\dfrac{180\times 180}{1{\cdot}85}} = \frac{247{\cdot}17}{1+1{\cdot}17} = \frac{247{\cdot}17}{2{\cdot}17}$$

$$= 113{\cdot}9 \text{ tons.}$$

The crippling load is 113·9 tons whereas it should have been $f_cA = 247{\cdot}17$ tons for pure compression.

$$\text{The safe load} = \frac{113{\cdot}9}{3} = 37{\cdot}97, \text{ say, } 38 \text{ tons.}$$

Example 4.

A cast iron column of hollow cylindrical section 16 feet long, with ends firmly built-in, has to carry an axial load of 30 tons. Determine the section, using Rankine's constants and a factor of safety of 8. Internal diameter to be $\frac{8}{10}$ of the external diameter.

Let D be the external diameter in inches.

The internal diameter $d = \cdot 8D$ inches.

The sectional area $A = \frac{\pi}{4}(D^2 - d^2) = \cdot 09\pi D^2$ in^2.

$$I = \frac{\pi}{64}(D^4 - d^4) = \frac{\pi}{4}(D^2 - d^2).k^2$$

$$\therefore \quad k^2 = \frac{D^2 + d^2}{16} = \frac{1\cdot 64 D^2}{16} \text{ in}^2.$$

The crippling load $P = 8 \times 30 - 240$ tons.

Using Rankine's formula,

$$P = \frac{f_c.A}{1 + \frac{a}{4}\cdot\frac{l^2}{k^2}}, \text{ where } f_c = 36 \text{ tons/in}^2., \text{ and } a = \frac{1}{1600}$$

we get,

$$240 = \frac{36 \times \cdot 09\pi D^2}{1 + \frac{1}{6400} \times \frac{192 \times 192 \times 16}{1\cdot 64 D^2}}$$

$$= \frac{36 \times \cdot 09\pi D^4}{D^2 + 56\cdot 2}$$

$$\therefore \quad D^4 = 23\cdot 58\ (D^2 + 56\cdot 2)$$

$$\therefore \quad (D^2 - 11\cdot 79)^2 = 1325 + 139$$

$$= 1464$$

$$\therefore \quad D^2 - 11\cdot 79 = 38\cdot 26$$

$$\therefore \quad D^2 = 50\cdot 05$$

and $D = 7\cdot 07''$, say $7''$.

The internal diameter $d = \cdot 8 \times 7 \doteq 5\cdot 6''$, say, $5\frac{1}{2}''$.

The thickness of metal will be $\frac{7 - 5\cdot 5}{2} = \frac{3}{4}''$.

Example 5.

A mild steel strut, 5 feet long, of angle-section $2\frac{1}{2}'' \times 2\frac{1}{2}'' \times \frac{1}{4}''$ *is fixed at one end and hinged at the other. Find by Rankine's formula the safe axial load for the strut, using a factor of safety of 3. For the* $2\frac{1}{2}'' \times 2\frac{1}{2}'' \times \frac{1}{4}''$ *angle, the area of section* $= 1{\cdot}19$ *in*^{2}*. Minimum radius of gyration* $= {\cdot}49''$*.*

$$\text{The crippling load } P = \frac{f_c . A}{1 + \frac{a}{2}\cdot\frac{l^2}{k^2}} = \frac{21 \times 1{\cdot}19}{1 + \frac{1}{15000} \times \frac{60^2}{{\cdot}49^2}}$$

$$= \frac{25}{1 + {\cdot}9995} = 12{\cdot}5 \text{ tons.}$$

$$\text{The safe axial load} = \frac{12{\cdot}5}{3} = 4{\cdot}17 \text{ tons.}$$

It should be noted that for an angle section, the least moment of inertia is not about the usual axes *XX* and *YY* but about an inclined axis about which it will bend. Structural steel tables therefore give the value of minimum radius of gyration for each angle section.

Example 6.

A steel stanchion is built-up of two $10'' \times 3\frac{1}{2}''$ *standard channels placed* $4\frac{1}{2}''$ *apart back to back with two* $12'' \times \frac{1}{2}''$ *plates riveted to each flange [fig. 88(c)]. If the ends are hinged, find the safe axial load for a length of 20 feet, using a factor of safety of 4. For the* $10'' \times 3\frac{1}{2}''$ *channel, the area of section* $= 7{\cdot}19$ *in*^{2}*.;* $I_{xx} = 109{\cdot}42$ *in*^{4}*.;* $I_{yy} = 7{\cdot}42$ *in*^{4}*.; distance of c.g. from the base* $= {\cdot}97''$*.*

Area of the built-up section

$$= 2 \times 7{\cdot}19 + 4 \times 12 \times \tfrac{1}{2} = 38{\cdot}38 \text{ in}^2.$$

$$I_{xx} = 2 \times 109{\cdot}42 + 2\left\{\tfrac{1}{12} \times 12 \times 1^3 + 12 \times (5{\cdot}5^2)\right\}$$

$$I_{yy} = 2 \times \tfrac{1}{12} \times 1 \times 12^3 + 2\left\{7{\cdot}42 + 7{\cdot}19\ (2{\cdot}25 + {\cdot}97)^2\right\}$$

$$= 288 + 14{\cdot}84 + 149{\cdot}10$$

$$= 451{\cdot}94 \text{ in}^4.$$

I_{yy} is the lesser of the two: Hence the required

$$k^2 = \frac{451{\cdot}94}{38{\cdot}38} = 11{\cdot}78 \text{ in}^2.$$

$$\therefore \quad \text{Crippling load } P = \frac{21 \times 38{\cdot}38}{1 + \dfrac{1}{7500} \times \dfrac{240^2}{11{\cdot}78}} = \frac{806}{1{\cdot}652}$$

$$= 487{\cdot}8 \text{ tons.}$$

$$\text{Safe load} = \frac{487{\cdot}8}{4} = 122 \text{ tons.}$$

Example 7.

A braced crane-jib is built-up of 4 — $3\frac{1}{2}'' \times 3\frac{1}{2}'' \times \frac{3}{8}''$ angles forming a square section 18″ overall as in fig. 88(j). If the length is 30 feet and the ends are considered hinged, calculate the safe axial load using Rankine's constants and a factor of safety of 3. For the $3\frac{1}{2}'' \times 3\frac{1}{2}'' \times \frac{3}{8}''$ angle, area of section = 2·49 in².; $J = 1''$, $I_{xx} = I_{yy} = 2{\cdot}80$ in⁴.; $k_{min} = {\cdot}68''$.

The moment of inertia of this section has been worked out in example 20 of chap. V. $I = 648{\cdot}64$ in⁴. and $k^2 = 65{\cdot}2$ in².

$$\therefore \quad \text{Crippling load } P = \frac{4 \times 2{\cdot}49 \times 21}{1 + \dfrac{1}{7500} \times \dfrac{360 \times 360}{65{\cdot}2}} = \frac{209{\cdot}16}{1{\cdot}265}$$

$$= 165{\cdot}3 \text{ tons.}$$

$$\text{Safe load} = \frac{165{\cdot}3}{3} = 55{\cdot}1 \text{ tons.}$$

Spacing of braces:

The lattice-bars used for bracing are usually flats about 2″ to 3″ wide and $\frac{5}{16}''$ to $\frac{3}{8}''$ thick. In regard to their spacing i.e. for the unsupported length l (see fig. 161), we may proceed thus:

Let L be the length of the built-up strut. If the section is properly braced, the load P coming on it will be shared equally by the four angles comprising the section and each angle will get $\frac{P}{4}$ as the axial load on it. Taking the end conditions for each individual angle as hinged at the rivets attaching the braces, the maximum unsupported length l so that the angle does not buckle under the load coming to its share will be given by:

$\frac{P}{4} = \frac{f_c.A}{1 + a\frac{l^2}{k^2}}$, where A is the area of the angle section and k is its minimum radius of gyration.

For the built-up section, as a whole,

$P = \frac{f_c \times 4A}{1 + a\frac{L^2}{K^2}}$, where K is the radius of gyration of the built-up section and the ends are considered hinged.

$$\therefore \quad \frac{f_c.A}{1 + a\frac{l^2}{k^2}} = \frac{f_c.A}{1 + a\frac{L^2}{K^2}}$$

or $$\frac{l}{k} = \frac{L}{K} \quad \text{i.e. } l = \frac{k}{K}.L.$$

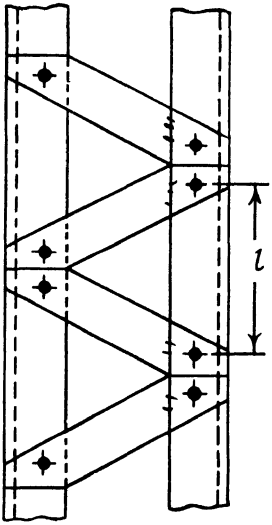


FIG. 161

If the end conditions of the built-up section are different, say both ends fixed, we have,

$$P = \frac{f_c \times 4A}{1 + \frac{a}{4} \times \frac{L^2}{K^2}}$$

Therefore,

$$\frac{f_c \times A}{1 + a\frac{l^2}{k^2}} = \frac{f_c \times A}{1 + \frac{a}{4}.\frac{L^2}{K^2}}$$

or $$\frac{l^2}{k^2} = \frac{L^2}{4K^2}$$

i.e. $$l = \frac{L}{2} \times \frac{k}{K}.$$

The maximum permissible spacing of braces in example 7 will be,

$$l = L\frac{k}{K} = 360 \times \frac{\cdot 68}{8{\cdot}07} = 30{\cdot}34'', \text{ say, } 2'\ 6''.$$

4. Design of struts and columns: From the previous examples it is clear that it is easy to apply the Rankine formula to work out the crippling or safe load for a strut or column of given section, length and end-conditions.

In designing, however, we know the length, the end-conditions and the safe load that has to be transmitted and we have to work out the necessary section from the Rankine formula $P = \dfrac{f.A}{1 + a.\dfrac{l^2}{k^2}}$ where P the load to be carried and f the working stress for the material are known. Until the section has been determined in shape and dimensions, its minimum radius of gyration is not known. Only in the case of a circular section, it is possible to make a straight use of the formula to determine the necessary diameter since, both A and k are functions of the diameter.

In all other cases, it is necessary to assume a reasonable value of the slenderness ratio $\frac{l}{k}$ and work out the safe load per unit area $f_r = \dfrac{f}{1 + a\dfrac{l^2}{k^2}}$.

Then the section area A required will be obtained from, $P = f_r \times A$

i.e. $A = \dfrac{P}{f_r}$.

Now, from the structural steel tables, select a suitable section whether it is I, L or T and check up the calculations by using the correct value of k for the selected section.

It is convenient to plot a graph of $\dfrac{P}{A}$ against $\dfrac{l}{k}$ as in fig. 162 and select the value of $\dfrac{P}{A}$ for an assumed slenderness ratio from the graph.

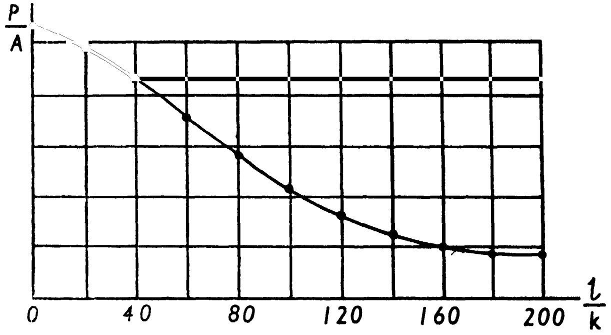


FIG. 162

5. Other empirical formulae: Johnson's parabolic formula: Straight line formula: For a short strut the safe axial load $P = f.A$, where f is the permissible compressive stress for the material and A is the area of section. Due to the possibility of buckling in struts which cannot be classified as short, the safe load will no longer be $f.A$ but less; in other words, the permissible load per unit area will have to be reduced from f to a smaller amount, depending upon the slenderness ratio $\dfrac{l}{k}$.

In the Rankine formula $\dfrac{P}{A} = \dfrac{f}{1 + a\dfrac{l^2}{k^2}}$, this reduction has been effected by dividing f (i.e. the permissible stress for a short strut) by $(1 + a\frac{l^2}{k^2})$, a factor evidently more than unity. Another way of effecting the reduction would be to subtract from f a suitable amount which is a function of the slenderness ratio. Thus,

$$\frac{P}{A} = f - \varphi\left(\frac{l}{k}\right)$$

If for $\varphi\left(\frac{l}{k}\right)$ we put $b\left(\frac{l^2}{k^2}\right)$ the expression becomes

$$\frac{P}{A} = f - b\left(\frac{l}{k}\right)^2 \text{ where } b \text{ is a constant.}$$

This is known as Johnson's parabolic formula, since, if $\frac{P}{A}$ is plotted against $\frac{l}{k}$, the curve is parabolic. The constants f and b in the formula being given, Johnson's formula is easier to work than Rankine's.

If the function $\varphi\left(\frac{l}{k}\right)$ is made simpler still as $c.\left(\frac{l}{k}\right)$ where c is a constant, the formula for $\frac{P}{A}$ becomes,

$$\frac{P}{A} = f - c.\frac{l}{k}.$$

If $\frac{P}{A}$ is plotted against $\frac{l}{k}$ it is a straight line relation and the formula is therefore known as the straight line formula. For mild steel, the constants are about 16000 lb/in^2. for f and about 60 to 80 for c for different end-conditions. Thus, the safe load in lb. per sq. inch $\frac{P}{A} = 16000 - 80\frac{l}{k}$ for both ends hinged and $\frac{P}{A} = 16000 - 60\frac{l}{k}$ for both ends fixed. If the loading is expressed in tons/in^2., $\frac{P}{A} = 7{\cdot}5 - {\cdot}025\frac{l}{k}$ for fixed ends, which indicates the value of the constants. The formula is valid for slenderness ratio $\not> 90$.

Mention may be made of Gordon's formula which is similar to Rankine's but simpler to use.

By this formula,

$P = \dfrac{f.A}{1 + a_1\dfrac{l^2}{b^2}}$, where b is the lesser overall dimension of the section and a_1 is a constant. If we re-arrange Rankine's formula as $P = \dfrac{f.A}{1 + \dfrac{ab^2}{k^2}.\dfrac{l^2}{b^2}}$ it will be seen that Gordon's constant a_1 is $\dfrac{a.b^2}{k^2}$ and is therefore not a mathematical constant like Rankine's. Its value will depend upon the material and the shape of the section.

6. Long columns under eccentric loading: When a short column is under an eccentric load P, the amount of eccentricity from the geometric axis being e, we have seen that the maximum intensity of compressive stress is given by,

$$p_{max} = p_o + p_b = \frac{P}{A} + \frac{P.e}{Z}.$$

Since, $Z = \dfrac{I}{y_c} = \dfrac{Ak^2}{y_c}$, we get,

$$p_{max} = \frac{P}{A} + \frac{P.ey_c}{Ak^2} = \frac{P}{A}\left(1 + \frac{ey_c}{k^2}\right).$$

If f is the permissible stress for the material, the safe load for a column of given section with a given eccentricity e is obtained from,

$$f = \frac{P}{A}\left(1 + \frac{ey_c}{k^2}\right)$$

i.e. $$P = \frac{f.A}{\left(1 + \frac{ey_c}{k^2}\right)}$$

If the buckling effect is taken into account we have seen that in the Rankine relation, the safe axial load is reduced by the factor $\dfrac{1}{1 + a\dfrac{l^2}{k^2}}$.

The Rankine formula for a column under eccentric loading may therefore be modified to take into account both eccentricity and buckling and be stated as:

$P = \dfrac{f.A}{(1 + e.\dfrac{y_c}{k^2})\ (1 + a\dfrac{l^2}{k^2})}$, if the column has both ends hinged. The value of the constant will be $\dfrac{a}{4}$ for both ends fixed and $\dfrac{a}{2}$ for one end fixed and the other hinged. If the value of f in the formula is the ultimate stress for the material, the corresponding value of P will be the crippling load.

The modified Rankine formula is empirical. We shall now investigate, from first principles, the value of the crippling load for a column of given length, section and end conditions and given eccentricity.

Consider the case of a column AB of length l with one end A fixed and the other end B free, on which the load is eccentric by e. Let P be the critical load at which the column deflects and let a be the deflection of the free end B. Consider a section X at a distance x from A and let y be its deflection.

The eccentricity of the load P on the section X is, therefore, $(a + e - y)$.

$$\therefore\quad EI\frac{d^2y}{dx^2} = P\ (a + e - y)$$

$$\therefore\quad \frac{d^2y}{dx^2} + \frac{P}{EI}.y = \frac{P(a + e)}{EI}.$$

Solving the equation,

$$y = C_1 \cos x\sqrt{\frac{P}{EI}} + C_2 \sin x\sqrt{\frac{P}{EI}} + (a + e)$$

And $$\frac{dy}{dx} = -C_1\sqrt{\frac{P}{EI}}\sin x\sqrt{\frac{P}{EI}} + C_2\sqrt{\frac{P}{EI}}\cos x\sqrt{\frac{P}{EI}}$$

At A, $x = 0$ and $y = 0$; also, $\frac{dy}{dx} = 0$

$\therefore \quad 0 = C_1 + (a + e)$

And $\quad 0 = C_2\sqrt{\frac{P}{EI}}$

$\therefore \quad C_2 = 0.$

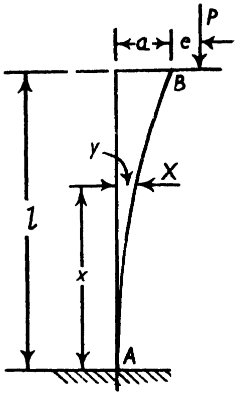


FIG. 163

The equation of the curve which the column takes up is, therefore,

$$y = (a + e)\left\{1 - \cos x\sqrt{\frac{P}{EI}}\right\}$$

At B, $x = l$ and $y = a$

$$\therefore \quad a = (a + e)\left\{1 - \cos l\sqrt{\frac{P}{EI}}\right\}$$

$$\therefore \quad (a + e)\cos l\sqrt{\frac{P}{EI}} = e$$

i.e. $$(a + e) = e\sec l\sqrt{\frac{P}{EI}}.$$

Of the sections of the column, that at A has the largest eccentricity of loading, viz., $(a + e)$ and has consequently the largest bending moment coming on it, viz., $P(a + e)$.

Since $(a + e) = e \sec l \sqrt{\frac{P}{EI}}$, the maximum B.M. to be resisted by the section is $P.e \sec l \sqrt{\frac{P}{EI}}$.

The maximum compressive stress induced in the section at A is, therefore, $p_o + p_b$

$$\text{or} \quad \frac{P}{A} + \frac{P.e \sec l\sqrt{\frac{P}{EI}}}{Z}.$$

In a short column with an eccentric load P, the bending moment at all sections is $P.e$. In the case of a long column the maximum bending moment is increased by the factor $\sec l \sqrt{\frac{P}{EI}}$ so that it is $P.e \sec l \sqrt{\frac{P}{EI}}$ instead of the usual $P.e$ for a short column.

If both ends are hinged, it can be similarly shown that the maximum bending moment is $P.e \sec \frac{l}{2}\sqrt{\frac{P}{EI}}$, while, for both ends fixed, the expression is $P.e. \sec \frac{l}{4}\sqrt{\frac{P}{EI}}$.

It is good to remember the expression for maximum bending moment for the standard case of both ends hinged which is $P.e. \sec \frac{l}{2}\sqrt{\frac{P}{EI}}$. It is then easy to put down the correct value for other end-conditions.

Example 8.

A C.I. column of hollow circular section, 7″ external diameter, thickness of metal $\frac{3}{4}$″, *length 16 feet has to take a load of 12 tons at an eccentricity of* $\frac{3}{4}$″ *from the geometrical axis. If the ends are fixed, calculate the maximum and minimum stress intensities induced in the section, taking* $E = 6000$ *tons/in*^{2}. *Also calculate the maximum permissible eccentricity so that no tension is induced anywhere in the section.*

The area of section $A = \frac{\pi}{4}(7^2 - 5{\cdot}5^2) = 14{\cdot}73 \text{ in}^2$.

$$I = \frac{\pi}{64}(7^4 - 5{\cdot}5^4) = 72{\cdot}93\ \text{in}^4.;\ k^2 = \tfrac{1}{16}\,(7^2 + 5{\cdot}5^2) = 4{\cdot}95\ \text{in}^2.$$

$$p_o = \frac{P}{A} = \frac{12}{14{\cdot}73} = {\cdot}815\ \text{ton/in}^2.;\ Z = \frac{I}{3{\cdot}5} = \frac{72{\cdot}93}{3{\cdot}5} = 20{\cdot}84\ \text{in}^3.$$

The maximum B.M. will be $P.e.$ sec $\frac{l}{4}\sqrt{\frac{P}{EI}}$

$$\text{Now, } \frac{l}{4}\sqrt{\frac{P}{EI}} = \frac{192}{4}\sqrt{\frac{12}{6000 \times 72{\cdot}93}} = 48 \times {\cdot}005236$$

$$= {\cdot}2513 \text{ radian or } 14{\cdot}40^\circ.$$

$$\sec 14{\cdot}4^\circ = 1{\cdot}032$$

$$\therefore \quad M = 12 \times \tfrac{3}{4} \times 1{\cdot}032 = 9{\cdot}288 \text{ ton-inches.}$$

$$\therefore \quad p_b = \frac{M}{Z} = \frac{9{\cdot}288}{20{\cdot}84} = {\cdot}446 \text{ ton/in}^2.$$

$$p_{max} = p_o + p_b = {\cdot}815 + {\cdot}446 = 1{\cdot}261 \text{ tons/in}^2.\text{, compressive.}$$

$$p_{min} = p_o - p_b = {\cdot}815 - {\cdot}446 = {\cdot}369 \text{ ton/in}^2.\text{, compressive.}$$

If tension is just on the point of being induced in the section, $p_b = p_o = {\cdot}815$ ton/in^2. The corresponding eccentricity will be,

$$e = \frac{{\cdot}815 \times {\cdot}75}{{\cdot}446} = 1{\cdot}37''.$$

Example 9.

The built-up stanchion in example 6, is subjected to a load of 80 tons. If the maximum permissible compressive stress is 5 tons/in^2., calculate the maximum permissible eccentricity of the load line from the axis YY of the section, the load line being in the vertical plane through the axis XX. $E = 13000$ tons/in^2.

The area of section $= 38{\cdot}38$ in^2.

$$I_{yy} = 451{\cdot}94\ \text{in}^4.;\ k^2 = \frac{451{\cdot}94}{38{\cdot}38} = 11{\cdot}78\ \text{in}^2.$$

$$Z_{yy} = \frac{451{\cdot}94}{6} = 75{\cdot}32\ \text{in}^3.$$

$$p_o = \frac{80}{38{\cdot}38} \quad 2{\cdot}084\ \text{tons/in}^2.$$

The maximum permissible bending stress

$$p_b = 5 - 2{\cdot}084 = 2{\cdot}916 \text{ tons/in}^2.$$

$$\therefore \qquad M_{max} = 2{\cdot}916 \times 75{\cdot}32 = 219{\cdot}7 \text{ ton-inches.}$$

$$\therefore \qquad P.e.\sec\frac{l}{2}\sqrt{\frac{P}{EI}} = 219{\cdot}7.$$

Now,

$$\frac{l}{2}\sqrt{\frac{P}{EI}} = 120\sqrt{\frac{80}{13000 \times 451{\cdot}94}} = 120 \times {\cdot}00369$$

$$= {\cdot}4428 \text{ radian or } 25°22'.$$

$$\therefore \qquad \sec\frac{l}{2}\sqrt{\frac{P}{EI}} = \sec 25°22' = 1{\cdot}107$$

$$\therefore \quad 80 \times e \times 1{\cdot}107 = 219{\cdot}7$$

and

$$e = 2{\cdot}48''.$$

From the examples worked out above, it is seen that if the column section, length and end-conditions are given, it is easy to work out the extreme stress-intensities due to a given load with a given eccentricity or to calculate the permissible eccentricity for a given load and a given permissible stress. However, if the safe load for a given section at a given stress and limit of eccentricity has to be determined, the necessary formula will have to be thrown into a workable form.

7. Prof. Perry's formula: Consider the standard case of a column with both ends hinged. Let p be the maximum permissible stress and e the eccentricity of loading.

Then,

$$p_{max} = p = p_o + p_b = \frac{P}{A} + \frac{P.e.\sec\frac{l}{2}\sqrt{\frac{P}{EI}}}{Z}$$

$$= \frac{P}{A} + \frac{P.e.y_c}{Ak^2}.\sec\frac{l}{2}\sqrt{\frac{P}{EI}}$$

$$= p_o\left(1 + \frac{e.y_c}{k^2}.\sec\frac{l}{2}\sqrt{\frac{P}{EI}}\right)$$

$$= p_o\left(1 + \frac{e.y_c}{k^2}.\sec\frac{\pi}{2}\sqrt{\frac{P}{P_e}}\right) \text{ where}$$

$$P_e = \frac{\pi^2 EI}{l^2}.$$

Prof. Perry found that the expression $\sec\frac{l}{2}\sqrt{\frac{P}{EI}}$ or $\sec\frac{\pi}{2}\sqrt{\frac{P}{P_e}}$ approximated very closely to $\frac{1{\cdot}2P_e}{P_e - P}$ or $\frac{1{\cdot}2p_e}{p_e - p_o}$

Substituting, we get,

$$p = p_o\left(1 + \frac{e.y_c}{k^2}\cdot\frac{1{\cdot}2p_e}{p_e - p_o}\right)$$

$$\therefore \quad \left(\frac{p}{p_o} - 1\right)\left(1 - \frac{p_o}{p_e}\right) = \frac{1{\cdot}2e.y_c}{k^2}.$$

This is Prof. Perry's approximate formula. In this form, it is easy to work out p_o for given values of p and e.

Example 10.

For the column in example 6, calculate the permissible load with an eccentricity of $1\frac{3}{4}$ inches from the axis YY if the maximum compressive stress is limited to 5 tons/in^2. E = 13000 tons/in^2.

$$P_e = \frac{\pi^2 EI}{l^2} = \frac{\pi^2 \times 13000 \times 11{\cdot}78 \times A}{240 \times 240}$$

$$\therefore \quad p_e = \frac{\pi^2 \times 13000 \times 11{\cdot}78}{240 \times 240} = 26{\cdot}24 \text{ tons/in}^2.$$

Substituting in Perry's formula,

$$\left(\frac{5}{p_o} - 1\right)\left(1 - \frac{p_o}{26{\cdot}24}\right) = \frac{1{\cdot}2 \times 1{\cdot}75 \times 6}{11{\cdot}78} = \frac{12{\cdot}6}{11{\cdot}78}$$

$$\therefore \quad (5 - p_o)\ (26{\cdot}24 - p_o) = \frac{26{\cdot}24 \times 12{\cdot}6}{11{\cdot}78}.p_o$$

$$= 28{\cdot}07\ p_o$$

$$\therefore \quad p_o^2 - 59{\cdot}31\ p_o + 131{\cdot}20 = 0$$

$$\therefore \quad (p_o - 29{\cdot}65)^2 = 879 - 131{\cdot}20 = 747{\cdot}80$$

$$\therefore \quad (p_o - 29{\cdot}65) = -27{\cdot}35$$

i.e. $p_o = 29{\cdot}65 - 27{\cdot}35 = 2{\cdot}3$ tons/in^2.

The safe load $P = 2{\cdot}3 \times 38{\cdot}38 = 88{\cdot}27$ tons.

8. Initial curvature on long column: Axial loading: Consider a column AB of length l hinged at both ends, with an initial curvature such that the central deflection is e'. Let y' be the deflection at a section X distant x from A. Taking the origin at A and assuming the initial shape of the column before loading as a sine curve, we have,

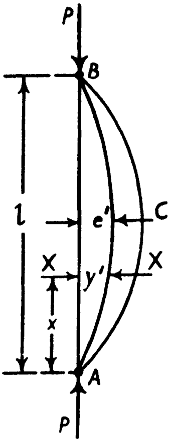


FIG. 164

$$y' = e' \sin \frac{\pi x}{l}.$$

This satisfies the initial conditions, since at A, $x = 0$ and $y' = 0$; at B, $x = l$, $y' = 0$; and at C, $x = \frac{l}{2}$, $y' = e'$.

Since $y' = e' \sin \frac{\pi x}{l}$

$$\frac{dy'}{dx} = \frac{\pi e'}{l} \cos \frac{\pi x}{l}$$

$$\text{and } \frac{d^2y'}{dx^2} = -\frac{\pi^2 e'}{l^2} \sin \frac{\pi x}{l} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

At the critical load P let the column deflect into the shape ACB (fig. 164). If y is now the deflection of the section X, the bending moment at the section is $P.y$ which brings about the change in the deflection from y' to y. We have, therefore,

$$EI\frac{d^2}{dx^2}(y-y') = -P.y.$$

Re-arranging and substituting for $\frac{d^2y'}{dx^2}$ from (1),

$$\frac{d^2y}{dx^2}+\frac{P}{EI}y = \frac{d^2y'}{dx^2} = -\frac{\pi^2e'}{l^2}\sin\frac{\pi x}{l} \quad\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Let the solution of this differential equation be:

$y = \lambda.e'\sin\frac{\pi x}{l}$, where λ is a constant of integration.

Then, $\frac{dy}{dx} = \frac{\lambda\pi}{l}.e'\cos\frac{\pi x}{l}$

and $\frac{d^2y}{dx^2} = -\frac{\lambda\pi^2}{l^2}.e'\sin\frac{\pi x}{l}$

To evaluate the constant λ substitute the expressions for y and $\frac{d^2y}{dx^2}$ in (2). We have,

$$-\frac{\lambda\pi^2}{l^2}.e'\sin\frac{\pi x}{l}+\frac{P}{EI}.\lambda e'\sin\frac{\pi x}{l} = -\frac{\pi^2}{l^2}.e'\sin\frac{\pi x}{l}$$

$$\therefore\quad \lambda\left\{\frac{\pi^2}{l^2}-\frac{P}{EI}\right\} = \frac{\pi^2}{l^2}$$

$$\text{or}\quad \lambda\left\{1-\frac{P.l^2}{\pi^2EI}\right\} = 1$$

$$\text{i.e.}\quad \lambda\left\{1-\frac{P}{P_e}\right\} = 1$$

$$\therefore\quad \lambda = \frac{P_e}{P_e-P}.$$

The equation of the curve of the deflected column is, therefore,

$$y = \frac{P_e}{P_e-P}.e'.\sin\frac{\pi x}{l} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

The maximum deflection y_c will occur at the central section C. Putting $x = \frac{l}{2}$,

$$y_c = \frac{P_e}{P_e-P}.e'.$$

The maximum bending moment will be at the centre, being,

$$P.y_c \text{ or } \frac{P.P_e}{P_e - P}.e'.$$

The maximum compressive stress p_{max} at the central section is given by,

$$p_{max} = p_o + p_b = \frac{P}{A} + \frac{M}{Z} = \frac{P}{A} + \frac{P.P_e}{(P_e - P)}.e'.\frac{y_c}{Ak^2}$$

$$= \frac{P}{A}\left\{1 + \frac{P_e}{P_e - P}.\frac{e'y_c}{k^2}\right\}$$

$$= p_o\left\{1 + \frac{p_e}{p_e - p_o}.\frac{e'y_c}{k^2}\right\}$$

or $$\left(\frac{p_{max}}{p_o} - 1\right)\left(1 - \frac{p_o}{p_e}\right) = \frac{e'y_c}{k^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (4)$$

9. Perry-Robertson formula: If a strut has initial curvature, the initial central deflection being e', the relation has worked out as in (4) above. For a straight column under eccentric loading, the amount of eccentricity being e, Perry's formula gave,

$$\left(\frac{p_{max}}{p_o} - 1\right)\left(1 - \frac{p_o}{p_e}\right) = \frac{1.2e.y_c}{k^2}.$$

The form of these formulæ—for initial curvature and eccentric loading—is exactly alike. Therefore, if the maximum deviation from straightness due to both causes, viz., initial curvature and eccentric loading be specified as e_1 being $(1{\cdot}2e + e')$, a column initially bent and also eccentrically loaded may be considered as equivalent to an axially loaded column with augmented initial curvature given by $e_1 = (1{\cdot}2e + e')$, for which, the maximum compressive stress induced is given by,

$$\left(\frac{p_{max}}{p_o} - 1\right)\left(1 - \frac{p_o}{p_e}\right) = \frac{e_1.y_c}{k^2}.$$

If p_{max} is allowed to go upto f, the permissible stress, substituting f for p_{max} and η for $\frac{e_1.y_c}{k^2}$, the expression becomes,

$$\left(\frac{f}{p_o}-1\right)\left(1-\frac{p_o}{p_e}\right)=\eta.$$

To obtain p_o, the permissible load per sq. inch of section, we may re-arrange the expression as under:

$$(f-p_o)\ (p_e-p_o)=\eta p_o p_e$$

or $p_o^2-p_o\{f+p_e\ (1+\eta)\}+f.p_e=0.$

Solving for p_o,

$$\left\{p_o-\frac{f+p_e\ (1+\eta)}{2}\right\}^2=\left\{\frac{f+p_e\ (1+\eta)}{2}\right\}^2-f.p_e$$

$$\therefore\quad p_o-\frac{f+p_e\ (1+\eta)}{2}=\pm\sqrt{\left\{\frac{f+p_e\ (1+\eta)}{2}\right\}^2-f.p_e}$$

$$\therefore\quad p_o=\frac{f+p_e\ (1+\eta)}{2}-\sqrt{\left\{\frac{f+p_e\ (1+\eta)}{2}\right\}^2-f.p_e}$$

where $\eta=\frac{e_1 y_c}{k^2}$ and $p_e=\frac{\pi^2 E}{\left(\frac{l}{k}\right)^2}$ for both ends hinged.

In Rankine's and other empirical formulæ, the column was expected to be perfectly straight and the loads to be truly axial, neither of which conditions can be assured in actual practice.

$p_o=\frac{f+p_e\ (1+\eta)}{2}-\sqrt{\left\{\frac{f+p_e\ (1+\eta)}{2}\right\}^2-f.p_e}$ is Prof. Perry's formula for the permissible load per square inch of section, allowing for such imperfections as initial crookedness, initial eccentricity of loading etc. In this formula, the term η which is $\frac{e_1 y_c}{k^2}$ is an unknown factor, depending as it does on e_1, the maximum deviation of the load line from the geometrical axis of the column. Prof. Andrew Robertson, on examining the available experimental observations, came to the conclusion that $\eta=\cdot 003\ (\frac{l}{k})$ represents well a very large number of experimental results. The British Standards Institution has adopted this value as a basis for calculating the safe loads for steel columns. The permissible loads per sq. inch for mild steel sections in the

British Standard Specifications have been determined by using the values of $f = 18$ tons/in²., $p_e = \dfrac{13000\pi^2}{\left(\dfrac{l}{k}\right)^2}$, $\eta = \cdot 003\dfrac{l}{k}$ and dividing the value of p_0 obtained from the Perry-Robertson formula by 2·36.

Example 11.

The cross-section of a steel stanchion is made up of a 10″ × 6″ × 40 lb. R.S. section with one 12″ × ½″ plate riveted to each flange. If the stanchion is 16 feet long, ends hinged, calculate the safe load it will carry, from the Perry-Robertson formula. The area of section for the 10″ × 6″ R.S.J. = 11·77 in².; $I_{xx} = 204\cdot 80$ in⁴.; $I_{yy} = 21\cdot 76$ in⁴.

The area of the built-up section $= 11\cdot 77 + 2 \times \frac{1}{2} \times 12$

$= 23\cdot 77$ in².

$$I_{yy} = 21\cdot 76 + 2 \times \tfrac{1}{12} \times \tfrac{1}{2} \times 12^3 = 165\cdot 76 \text{ in}^4.$$

$$\therefore \quad k^2 = \frac{165\cdot 76}{23\cdot 77} = 6\cdot 974 \text{ in}^2.$$

$$\therefore \quad k = 2\cdot 64''.$$

Now $l = 192''$, $\therefore \quad \dfrac{l}{k} = \dfrac{192}{2\cdot 64} = 73.$

In the Perry-Robertson formula,

$$\eta = \cdot 003 \times \frac{l}{k} = \cdot 003 \times 73 = \cdot 219.$$

$$p_e = \frac{13000\pi^2}{73^2} = 24\cdot 08 \text{ tons/in}^2.$$

$$\frac{f + p_e(1+\eta)}{2} = \frac{18 + 24\cdot 08 \times 1\cdot 219}{2} = \frac{18 + 29\cdot 36}{2}$$

$$= 23\cdot 68$$

$$\therefore \quad p_0 = \frac{f + p_e(1+\eta)}{2} - \sqrt{\left\{\frac{f + p_e(1+\eta)}{2}\right\}^2 - f.p_e}$$

$$= 23\cdot 68 - \sqrt{(23\cdot 68)^2 - 18 \times 24\cdot 08}$$

$$= 23{\cdot}68 - 11{\cdot}29 = 12{\cdot}39 \text{ tons/in}^2.$$

$$\therefore\ P = p_0.A = 12{\cdot}39 \times 23{\cdot}77 = 294{\cdot}5 \text{ tons.}$$

Dividing by 2·36, the safe load $= \frac{294{\cdot}5}{2{\cdot}36} = 124{\cdot}8$ tons.

Note: B. S. 449 now uses $f = 15{\cdot}25$ tons/in^2 and a load factor of 2 against 2·36.

10. Struts with transverse loading: If a strut, besides carrying an axial load, has also loads acting on it at right angles to its axis, the section will have to resist a direct thrust and a bending moment due to the transverse loading. If the strut is long, the direct thrust which may be truly axial at the ends, will itself impose an additional bending moment at a cross-section, being eccentric to the extent of the deflection of the section under the transverse loading. The bending stresses at the section will be the algebraic sum of those produced by transverse loads and those produced by the eccentricity of the longitudinal load.

We shall consider the case of a strut, of length l with hinged ends, carrying a uniformly distributed load of w per unit run transversely. Let the longitudinal thrust be P.

The maximum bending moment due to longitudinal and transverse loading will be at the centre and may be put down as:

$$M = \frac{wl^2}{8} + P.y_c \text{ approximately, where } y_c = \frac{5}{384}\cdot\frac{wl^4}{EI}.$$

The extreme intensities of stress at the section will be given by,

$$p_{max} = p_o + p_b = \frac{P}{A} + \frac{M}{Z_c}$$

$$\text{and } p_{min} = p_o - p_b = \frac{P}{A} - \frac{M}{Z_t}.$$

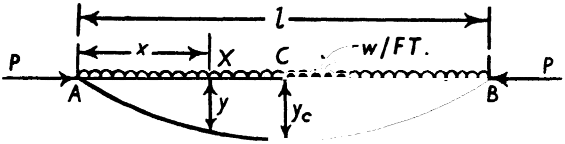


FIG. 165

Proceeding from first principles, consider the strut in fig. 165 carrying an axial load P at the ends and a uniformly distributed transverse load w per unit run across its entire length l. Taking the origin at A, the bending moment at a section X at a distance x from A is given by,

$$EI\frac{d^2y}{dx^2} = -\frac{wl}{2}.x + \frac{wx^2}{2} - P.y, \; y \text{ being the deflection at } X.$$

$$\therefore \quad \frac{d^2y}{dx^2} + \frac{P}{EI}.y = -\frac{wx\ (l-x)}{2EI}$$

Solving the differential equation,

$$y = C_1 \cos x\sqrt{\frac{P}{EI}} + C_2 \sin x\sqrt{\frac{P}{EI}} - \frac{wx\ (l-x)}{2P} - \frac{wEI}{P^2}$$

At A, $x = 0$, $y = 0$

$$\therefore \quad 0 = C_1 - \frac{wEI}{P^2} \text{ or } C_1 = \frac{wEI}{P^2}$$

At C, $x = \frac{l}{2}$ and $\frac{dy}{dx} = 0$

$$\therefore \quad 0 = -C_1\sqrt{\frac{P}{EI}} \sin\frac{l}{2}\sqrt{\frac{P}{EI}} + C_2\sqrt{\frac{P}{EI}} \cos\frac{l}{2}\sqrt{\frac{P}{EI}}$$

$$\therefore \quad C_2 = C_1 \tan\frac{l}{2}\sqrt{\frac{P}{EI}}$$

Substituting for C_1 and C_2, the expression becomes,

$$y = \frac{wEI}{P^2}\left\{\cos x\sqrt{\frac{P}{EI}} + \tan\frac{l}{2}\sqrt{\frac{P}{EI}} \sin x\sqrt{\frac{P}{EI}}\right\}$$
$$- \frac{wx\ (l-x)}{2P} - \frac{wEI}{P^2}$$

$\therefore$ At the centre,

$$y_c = \frac{wEI}{P^2}\left\{\cos\frac{l}{2}\sqrt{\frac{P}{EI}} + \tan\frac{l}{2}\sqrt{\frac{P}{EI}} . \sin\frac{l}{2}\sqrt{\frac{P}{EI}}\right\}$$
$$- \frac{wl^2}{8P} - \frac{wEI}{P^2}$$
$$= \frac{wEI}{P^2}\left\{\sec\frac{l}{2}\sqrt{\frac{P}{EI}} - 1\right\} - \frac{wl^2}{8P}.$$

The maximum B.M. due to both the transverse and longitudinal loading occurs at C, being,

$$M_c = -\frac{wl^2}{8} - P.y_c$$

$$= -\frac{wl^2}{8} - P\left\{\frac{wEI}{P^2}\left(\sec\frac{l}{2}\sqrt{\frac{P}{EI}} - 1\right) - \frac{wl^2}{8P}\right\}$$

$$= -\frac{wEI}{P}\left\{\sec\frac{l}{2}\sqrt{\frac{P}{EI}} - 1\right\}$$

Using the expansion of $\sec\theta$, viz.,

$$\sec\theta = 1 + \frac{\theta^2}{2!} + \frac{5\theta^4}{4!} + \frac{61\theta^6}{6!} \dots\dots\dots\dots\dots\dots\dots\dots$$

$$M_C = -\frac{wEI}{P}\left\{\frac{l^2}{8}.\frac{P}{EI} + \frac{5l^4}{384}.\frac{P^2}{E^2I^2} + \frac{61l^6}{720 \times 64} \times \frac{P^3}{E^3I^3} + .\right\}$$

$$= -\left\{\frac{wl^2}{8} + \frac{5}{384}.\frac{wl^4}{EI}.P\right\}$$ approximately, neglecting other terms. It will be seen that the first term represents the B.M. due to the transverse loading and the second term, the B.M. due to the longitudinal load being eccentric to the extent of the central deflection under transverse loading.

If the strut has a concentrated load W acting transversely at the middle, we may similarly state that, approximately,

$$-M_c = \frac{Wl}{4} + P.y_c, \text{ where } y_c = \frac{Wl^3}{48EI}.$$

EXAMPLES IX

(1) A mild steel tube, 1″ external diameter, ⅛″ thick, is 10 feet long and used as a strut, free at both ends. Calculate the collapsing load by Euler's formula. $E = 13000$ tons/in². (·3 ton.)

(2) Find the crushing load by Rankine's formula for a hollow C.I. column of 12″ external diameter, thickness of metal 1½″ and 25 feet long, if both ends are fixed. (891 tons.)

(3) A 10″×6″×40 lb. R.S. joist is used as a strut, 20 feet long, one end fixed, the other hinged. Calculate the crippling load by Rankine's formula. Compare this with the load obtained by the

Euler formula, taking $E = 13200$ tons/in^2. For what length of this strut will the two formulae give the same crippling load? For the joist, area of section $= 11{\cdot}77$ in^2.; $I_{xx} = 204{\cdot}80$ in^4.; $I_{yy} = 21{\cdot}76$ in^4. (80·26 tons; 101 tons; 33 feet.)

(4) A strut, 18 feet long, ends free, has to be erected to withstand a load of 40 tons. Two sections are available: (i) A $12'' \times 6''$ R.S.J., thickness of flanges ·72″, thickness of web ·4″. (ii) A C.I. section, 9″ external diameter, thickness of metal 1″. Which is more suitable? Use Rankine's constants and a factor of safety of 3 in both cases. (The C.I. section.)

(5) A $10'' \times 8''$ R.S. section, flanges $\frac{3}{4}''$ thick, web $\frac{1}{2}''$ thick, is used as a strut, 16 feet long, both ends fixed. Estimate the safe axial load that it will carry, using Rankine's constants and a factor of safety of 4. (65 tons.)

(6) Find by Rankine's formula the safe axial load which an angle iron strut $2\frac{1}{2}'' \times 2\frac{1}{2}'' \times \frac{5}{16}''$, 6 feet long, one end fixed, the other hinged, will carry, using a factor of safety of 3. For the angle, area of section $= 1{\cdot}46$ in^2.; minimum radius of gyration $= {\cdot}48''$. (4·09 tons.)

(7) A $4'' \times 3'' \times \frac{1}{2}''$ Tee is used as a strut, 8 feet long, ends hinged. Calculate the safe axial load by Rankine's formula, using a factor of safety of 3. (8·6 tons.)

(8) A C.I. column of hollow circular section has to transmit a load of 80 tons. Calculate the external diameter if the column is 22 feet long, both ends fixed. The thickness of metal is to be $1\frac{1}{2}''$. Use Rankine's constants and a factor of safety of 8. (10″.)

(9) A mild steel stanchion is built-up of one $6'' \times 5'' \times 25$ lb. R.S.J. with one $6'' \times \frac{1}{2}''$ plate riveted to each flange. If it is 12 feet long, with one end fixed and the other hinged, calculate the safe axial load by Rankine's formula using a factor of safety of 4. For the joist, area of section $= 7{\cdot}35$ in^2.; $I_{xx} = 43{\cdot}61$ in^4.; $I_{yy} = 9{\cdot}116$ in^4. (41·7 tons.)

(10) A stanchion is built-up of two $10'' \times 5'' \times 30$ lb. R.S. Joists placed 6″ centre to centre with two $12'' \times \frac{1}{2}''$ plates riveted to each flange. If it is 20 feet long, both ends fixed, calculate the safe axial load using Rankine's constants and a factor of safety of 3. For each joist, area of section $= 8{\cdot}82$ in^2.; $I_{xx} = 145{\cdot}6$ in^4.; $I_{yy} = 9{\cdot}79$ in^4. (248·9 tons.)

(11) A strut is built-up of two $6'' \times 3'' \times \frac{3}{8}''$ Tee sections riveted back to back on the 6″ sides so as to form the section of a cross. Use Gordon's formula to determine the safe load for the strut, length 12 feet, ends fixed. The constants may be taken as 21 tons/in^2. and $\frac{1}{250}$ for hinged ends. Use a factor of safety of 3. (28·7 tons.)

(12) A double angle strut is made up of two $5'' \times 3'' \times \frac{3}{8}''$ angles riveted together along the longer legs with $\frac{1}{2}''$ separators in between. Calculate the safe axial load for the strut, 8 feet long, ends fixed, the safe load in lb. per sq. inch being $16000 - \frac{1}{3}\left(\frac{l}{k}\right)^2$ for fixed ends. For each angle area of section $= 2{\cdot}86$ in^2; $\bar{x} = {\cdot}693''$; $\bar{y} = 1{\cdot}68''$; $I_{xx} = 7{\cdot}24$ in^4.; $I_{yy} = 1{\cdot}97$ in^4. (36 tons.)

(13) A stanchion has to transmit an axial load of 35 tons. The length is 14 feet, ends hinged. Select a suitable R.S. section, using Rankine's formula with a factor of safety of 3.
($12'' \times 6'' \times 54$ lb.)

(14) A stanchion is built-up of a $15'' \times 6'' \times 59$ lb. R.S.J. with one $8'' \times 6'' \times 35$ lb. R.S.J. riveted symmetrically on each side of the web to form the section of a cross. For the smaller joist, area of section $= 10{\cdot}29$ in^2.; $I_{xx} = 110{\cdot}5$ in^4.; $I_{yy} = 17{\cdot}95$ in^4.; for the bigger joist, area of section $= 17{\cdot}35$ in^2.; $I_{xx} = 628{\cdot}9$ in^4; $I_{yy} = 28{\cdot}22$ in^4.; thickness of web $\frac{1}{2}''$. Calculate the safe axial load for this stanchion of 25 feet length, ends fixed, using the safe load in tons per sq. inch at $(7{\cdot}5 - {\cdot}025\,\frac{l}{k})$ for fixed ends. (214·1 tons.)

(15) The stanchion in question 10, transmits a load of 150 tons with an eccentricity of 2″ from the axis *YY*, the load line being in the vertical plane through the axis *XX*. Calculate the extreme intensities of stress induced. $E = 13000$ tons/in^2.
(7·64 tons/in^2., compr.; ·435 tons/in^2., tensile.)

(16) A stanchion, 18 feet long, ends free, is built-up of two $15'' \times 4''$ standard channels placed 6″ back to back, with one $16'' \times \frac{1}{2}''$ plate riveted to each flange. It carries a load of 130 tons, which is off the axis *YY*, in the vertical plane through the axis *XX*. Calculate the permissible eccentricity if the maximum permissible compressive stress is 7 tons/in^2. For each channel, area of section $= 12{\cdot}33$ in^2.; distance of c.g. from base $= {\cdot}935''$; $I_{xx} = 377$ in^4.; $I_{yy} = 14{\cdot}55$ in^4.; $E = 13000$ tons/in^2. (2·54″.)

(17) In the preceding question, if the maximum eccentricity is limited to 2″, estimate the safe load for the stanchion if the maximum compressive stress is not to exceed 7 tons/in^2. (135·0 tons.)

(18) A stanchion is built-up of an $8'' \times 6'' \times 35$ lb. R.S. section with a $9'' \times \frac{1}{2}''$ plate riveted to each flange. Estimate the safe load for this stanchion, length 14 feet, ends hinged, from the Perry-Robertson formula, using the constants specified by the B.S.S. For the joist, area of section $= 10{\cdot}30$ in^2; $I_{xx} = 115{\cdot}06$ in^4; $I_{yy} = 19{\cdot}54$ in^4. (91·5 tons.)

(19) In an experimental determination of the buckling load for $\frac{1}{2}''$ diameter mild steel pin-ended struts of various lengths, two of the values obtained were: (i) length 20″, load 0·98 ton (ii) length 8″, load 2·65 tons.

(a) Make the necessary calculations and then state whether either of the values conforms with the Euler formula for the critical load.

(b) Assuming that both values are in agreement with the Rankine formula, find the two constants for this formula. Take $E = 13200$ tons/in^2. (L. U.)

$$\left\{ \text{(a) Yes for (i); No for (ii); (b) } f_c = 20 \text{ tons/in}^2;\ a = \frac{1}{8523} \right\}$$

(20) A bar of circular cross-section, 10 feet long, weighing 15 lb., initially straight, is freely supported at its ends in a horizontal position. The central deflection due to its own weight is 0·72″. If this bar is used as a vertical strut, position and direction fixed at both ends and loaded axially, find the theoretical buckling load by Euler's formula. (L. U.)

(1285 lb-wt.)

(21) A vertical stanchion 20 feet long consists of an *I*-section 8″ × 6″ ($A = 10{\cdot}3$ in^2.; $I_{xx} = 115{\cdot}1$ in^4.). An eccentric vertical load P is applied to the stanchion, its line of action being in the plane of *yy*-axis and at an eccentricity of 6″ with respect to the *xx*-axis. Treating the ends as hinged, find the value of the load P so that the maximum stress in the stanchion is 8 tons/in^2.,

(a) neglecting deflection;

(b) taking deflection into account and proving any formula used. $E = 13500$ tons/in^2. (L. U.)

{(a) 26·19 tons; (b) 21·73 tons.}

(22) Derive a formula for the maximum compressive stress induced in an initially straight slender, uniform strut when loaded along an axis having an eccentricity e at both ends which are pin-jointed.

A straight steel pin-jointed strut is 2″ in diameter and 50″ long. Calculate: (a) the Euler crippling load when loaded along the central axis,

(b) the eccentricity which will cause failure at 75% of this load if the yield point stress of the material is 18 tons/in^2. Take $E = 13400$ tons/in^2. (L. U.)

{(a) 41·57 tons; (b) $e = {\cdot}0423''$.}

(23) Two 10″×3″ mild steel channels are welded together at their toes to form a box section 10″×6″×20 feet long. The beam is hinged at its ends and carries a uniformly distributed load of 0·2 ton per foot run of a 6″ side and an axial end thrust of 30 tons. If the beam is prevented from bending in a lateral direction, find the maximum deflection and the maximum stress in the material. Area of beam section = 11·34 in².; I of section about axis of bending = 165·3 in⁴. $E = 13200$ tons/in². (L. U.)

(·3639″; 6·6 tons/in².)

(24) A thin vertical strut of uniform cross-section and length l is rigidly fixed at the bottom, and the top end is free. At the top, there is a horizontal load H and a vertical downward load W acting through the centroid of the section. Prove that the horizontal deflection at the top is $\frac{H}{W}\left(\frac{\tan \mu l}{\mu} - l\right)$ where $\mu^2 = \frac{W}{EI}$. (L. U.)

(25) An initially straight strut, l inches long, has lateral loading w per inch and a longitudinal load P applied with eccentricity e inch at both ends. If the strut has area A, relevant second moment of area I and section modulus Z and the end moments and lateral loading have opposing effects, find an expression for the central bending moment and show that the maximum stress at the centre will be equal to

$$\frac{P}{A} + \frac{\left(Pe - \frac{wEI}{P}\right) \sec \frac{l}{2} \sqrt{\frac{P}{EI}} + \frac{wEI}{P}}{Z} \qquad \text{(L. U.)}$$

CHAPTER X

RADIAL PRESSURE — CYLINDRICAL AND SPHERICAL SHELLS

1. Thin seamless cylindrical shells: Consider a thin cylindrical shell of internal diameter d, thickness of metal t and length l, subjected to an internal pressure of intensity p. Consider a longitudinal section XX of the shell through its axis, dividing the shell into two portions A and B as shown in fig. 166(a). Now consider two elementary strips of the shell subtending an angle $\delta\theta$ at the centre, at an angle θ on either side of the vertical through the centre O. The effect of the radial intensity of pressure p on each strip is a normal force $\delta P_n = p \times r\delta\theta \times l$, where r is the radius of the shell. The resultant of these two normal forces is, evidently, $\delta P = 2\delta P_n \cos\theta = 2p.r\delta\theta.l.\cos\theta$, acting along the vertical centre line in fig. 166(a). The total effect of the radial pressure on the portion A or B of the shell is, therefore, a single bursting force $P = \sum_0^{\frac{\pi}{2}} 2prd\theta.l\cos\theta$

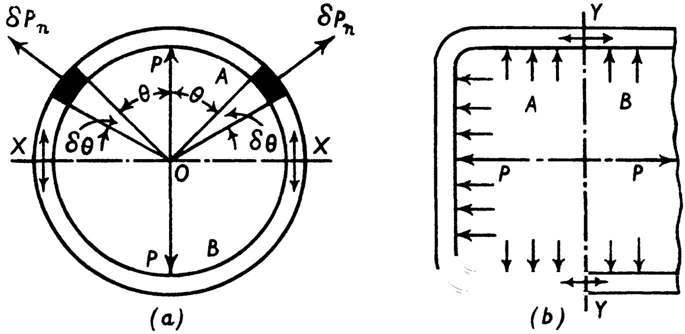


Fig. 166

$$= 2prl\int_0^{\frac{\pi}{2}} \cos\theta.d\theta = 2p.r.l = p.d.l.$$

The bursting force P is given by the product of the intensity of radial pressure and the projected area.

If f_1 is the uniform intensity of stress set up across the section XX, the resisting force $= f_1 \times 2l.t$, $2l.t$ being the area of the resisting section.

Equating the bursting force on A or B to the resisting force set up at the section, we have,

$$p.d.l = f_1 \times 2l.t$$

$\therefore \ f_1 = \frac{pd}{2t}$, which is a wholly tensile stress across the section XX. If we take any other longitudinal section through the axis of the shell, the result will be the same, viz., a wholly normal stress f_1 across the section of intensity $\frac{pd}{2t}$. Because the direction of this stress is along the circumference of the shell, it is known as the circumferential or hoop stress.

Next consider a section YY of the shell at right angles to its axis, dividing the shell into two parts A and B as in fig. 166(*b*). The effect of the radial pressure p on the portion A of the shell is a bursting force $P = p \times \frac{\pi}{4}d^2$, the effect of the pressure p bearing on the cylindrical portion being self-balanced. If f_2 is the uniform intensity of stress set up by the section YY to resist this bursting force, the resisting force $= f_2 \times \pi d.t$, $\pi d.t$ being the area of the resisting section.
Equating,

$$p \times \frac{\pi}{4}d^2 = f_2 \times \pi dt$$

$\therefore \quad f_2 = \frac{pd}{4t}$, which is a wholly tensile stress across the section YY. If we take any other circumferential section at right angles to the axis of the shell, the result will be the same, viz., a wholly normal stress f_2 across the section of intensity $\frac{pd}{4t}$. Because the direction of this stress is along the length of the shell it is known as the longitudinal stress. It may be noted that its intensity is one-half of the intensity of circumferential or hoop stress.

At any point in the metal of the shell, then, the circumferential stress f_1 across the longitudinal section is $\frac{pd}{2t}$, while the longitudinal stress f_2 across the circumferential section is $\frac{pd}{4t}$. These are both tensile and being wholly normal stresses are the principal stresses at the point, the planes across which they act, being the principal planes.

Since we have seized upon the principal planes and principal stresses at any point in the metal of the shell, we can work out any desired result without difficulty. For instance, the maximum intensity of shear stress at the point will be $\frac{f_1 - f_2}{2}$, i.e. $\frac{pd}{8t}$ on diagonal planes mutually at right angles and inclined at 45° to the principal planes.

The circumferential strain e_1 which is the major principal strain will be given by,

$$e_1 = \frac{f_1}{E} - \frac{f_2}{mE} = \frac{f_1}{E}\left(1 - \frac{1}{2m}\right) = \frac{pd}{2tE}\left(1 - \frac{1}{2m}\right).$$

Similarly, the longitudinal strain e_2 is given by,

$$e_2 = \frac{f_2}{E} - \frac{f_1}{mE} = \frac{f_1}{E}\left(\tfrac{1}{2} - \frac{1}{m}\right) = \frac{pd}{2tE}\left(\tfrac{1}{2} - \frac{1}{m}\right).$$

The changes in the dimensions of the shell may now be worked out.

$$\text{Circumferential strain} = \frac{\text{change of circumference}}{\text{original circumference}}$$

$$= \frac{\pi . \delta d}{\pi . d} = \frac{\delta d}{d} = e_1,$$ from which the change of diameter δd may be evaluated.

Similarly, the longitudinal strain $= \frac{\delta l}{l} = e_2$ from which the change of length δl may be obtained. For the change in volume,

$$V = \frac{\pi}{4} d^2 . l$$

$$\therefore \quad \delta V = \frac{\pi}{4}d^2.\delta l + 2 \times \frac{\pi}{4}d.\delta d.l$$

$$\therefore \quad \frac{\delta V}{V} = \frac{\delta l}{l} + \frac{2\delta d}{d} = e_2 + 2e_1.$$

The relation enables us to work out the change of volume δV that takes place under the radial pressure p.

If f is the permissible tensile stress for the material of which the shell is made, the major principal stress f_1 must not exceed the working stress f.

$$\therefore \quad \frac{pd}{2t} \leqq f.$$

If the thickness of the shell has to be designed for a given working pressure in the shell at a stress not exceeding a given working stress for the material, the necessary thickness will be obtained from the above relation, viz.,

$$t \geqq \frac{pd}{2f}.$$

Example 1.

Calculate the necessary thickness of metal for a seamless pipe, 2 feet in diameter, to carry fluid under a pressure of 250 lb/in^2., at a stress not exceeding 12000 lb/in^2.

$$t \geqq \frac{pd}{2f} \geqq \frac{250 \times 24}{2 \times 12000} \geqq \tfrac{1}{4}''.$$

Example 2.

A cylindrical shell, 8 feet long, 3 feet in diameter, thickness of metal $\frac{1}{2}''$ is subjected to an internal pressure of 200 lb/in^2. Calculate the maximum intensity of shear stress induced and also the changes in the dimensions of the shell. Given $E = 30 \times 10^6$ lb/in^2. and Poisson's ratio = ·3.

$$\text{Circumferential stress } f_1 = \frac{pd}{2t} = \frac{200 \times 36}{2 \times \frac{1}{2}} = 7200 \text{ lb/in}^2.$$

$$\text{Longitudinal stress } f_2 = \frac{pd}{4t} = 3600 \text{ lb/in}^2.$$

(15) Calculate the thickness of the shell of a bomb calorimeter of spherical form, 4″ internal diameter, if the allowable working stress is 4 tons/in^2., and internal pressure is 2 tons/in^2. (C. U.)

(·52″.)

(16) A thick-walled cylinder 8″ internal diameter is to contain fluid at a pressure of 3·5 tons/in^2. Find the necessary thickness if the maximum *shearing* stress is not to exceed 6 tons/in^2. What will then be the maximum and minimum values of the hoop stress in the material?

If the inner surface becomes corroded and the cylinder has to be re-bored, by how much can the inside diameter be increased without raising by more than 5% the maximum shearing stress induced by the same internal pressure? (L. U.)

$$\left\{\begin{array}{ll}\text{(i)} & t = 2{\cdot}2''; \ 8{\cdot}5 \text{ tons/in}^2; \ 5 \text{ tons/in}^2. \\ \text{(ii)} & {\cdot}262''.\end{array}\right\}$$

(17) A cylindrical thin-walled vessel with closed ends made of an alloy has an internal diameter of 8″ and a wall thickness of $\frac{1}{4}$″. The cylinder is strengthened by surrounding it with a single layer of steel wire of diameter $\frac{1}{16}$″ closely wound under tension. Determine the minimum tension under which the wire must be wound if the hoop tension in the cylinder is not to exceed 7000 lb/in^2. when the cylinder is subjected to an internal pressure of 750 lb/in^2. Find also the final tension in the wire. E for steel = 13500 tons/in^2.; E for the alloy = 6750 tons/in^2., and $\frac{1}{m} = 0{\cdot}33$. (L. U.)

(11110 lb/in^2.; 25470 lb/in^2.)

(18) A long copper cylinder with closed ends, 5$\frac{3}{4}$″ external diameter and wall thickness $\frac{1}{8}$″, is to be reinforced by winding on a single close layer of steel wire 0·064″ diameter. Calculate the necessary winding tension in the wire if an internal gauge pressure of 1500 lb/in^2. in the cylinder is to give a tensile stress of 6 tons/in^2. in the cylinder walls. Determine also the final stress in the wire. Take E for copper as 0·6 times that for steel and Poisson's ratio 0·3. (L. U.)

(20640 lb/in^2.; 48640 lb/in^2.)

(19) A cast iron cylinder has outside and inside diameters of 8″ and 5″. If the ultimate tensile strength of cast iron is 10 tons/in^2., find, according to each of the following theories of failure, the internal pressure which would cause rupture:

(a) maximum principal stress theory, (b) maximum strain theory, (c) maximum strain energy theory.

Assume no longitudinal stress in the cylinder. Which of the results obtained, do you consider should be applied to this case? Take Poisson's ratio $= \frac{1}{4}$. (L. U.)

{(a) 4·382 tons/in^2.; (b) 3·95 tons/in^2.; (c) 3·687 tons/in^2.}

(20) A steel cylinder of outside diameter 12″ and inside diameter 10″ is shrunk on to one having diameters 10″ and 8″, the interference fit being such that under an internal pressure p, the inner tensile stress in both cylinders = 12200 lb/in^2.

Find the initial difference in the nominal 10″ diameters and the value of p if $E = 30 \times 10^6$ lb/in^2. (L. U.)

{0·001054″; 5360 lbs/in^2.}

(21) A compound steel cylinder has a bore of 4″ and an outside diameter of 8″, the diameter at the common surface being 6″. Find the radial pressure at the common surface which must be provided by shrinkage if the resultant maximum hoop tension in the inner cylinder under a superimposed internal pressure of 4 tons/in^2. is to be half the value of the maximum hoop tension which would be produced in the inner cylinder if that cylinder alone were subjected to an internal pressure of 4 tons/in^2.

Determine the final hoop tensions at the inner and outer surfaces of both cylinders under the internal pressure of 4 tons/in^2. and sketch a graph to show how the hoop tension varies across the cylinder wall. (L. U.)

Radial pressure at 3″ radius = 0·41 ton/in^2.

Final stresses:

Inner tube		Outer tube	
$x = 2''$	$x = 3''$	$x = 3''$	$x = 4''$
5·2 tons/in^2.	2·64 tons/in^2.	5·16 tons/in^2.	3·72 tons/in^2.

CHAPTER XI

RIVETED JOINTS

1. Introductory: In the study of thin cylindrical shells, it was mentioned that a boiler shell of requisite capacity has to be built-up by jointing plates longitudinally and circumferentially. We use a joint only when we must, the effect of jointing being necessarily to weaken the plate. In structural steel work as in roof-trusses and bridge-trusses, joints are necessary to connect members. Rivets may be used for the joints which may also be effected by welding. We shall now investigate the strength of riveted joints, bearing in mind the fact that it is extremely difficult to gauge accurately the stress-distributions in a riveted joint. Theoretical considerations, however, are a useful guide and will be studied as such.

2. Forms of riveted joints: Fig. 174 shows the usual forms of riveted joints. In (*a*) and (*b*), the plates to be jointed are lapped and riveted. In (*c*) and (*d*), they butt against each other and are connected either by a single cover on one side or by two cover plates one on each side. Fig. 174(*e*) shows chain-riveting while fig. 174(*f*) shows zigzag riveting where the spacing of rivets is staggered.

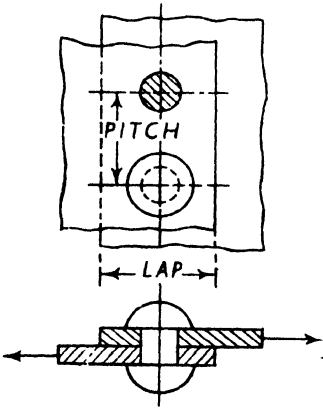


FIG. 174(*a*)

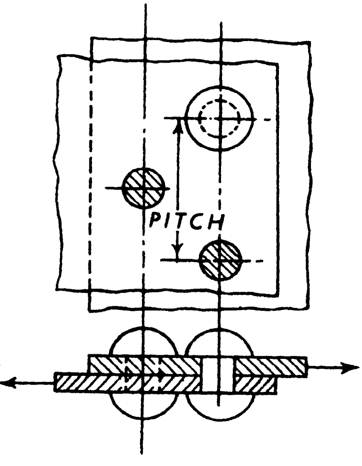
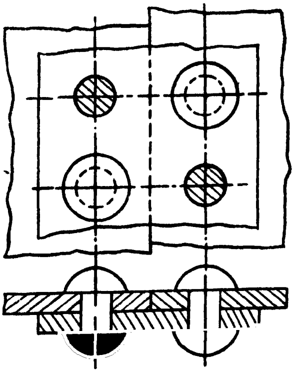
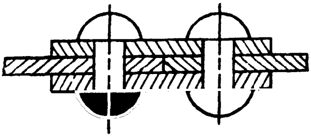


FIG. 174(*b*)

Pitch of rivets:



SINGLE COVER



DOUBLE COVERS

Fig. 174(*c*)

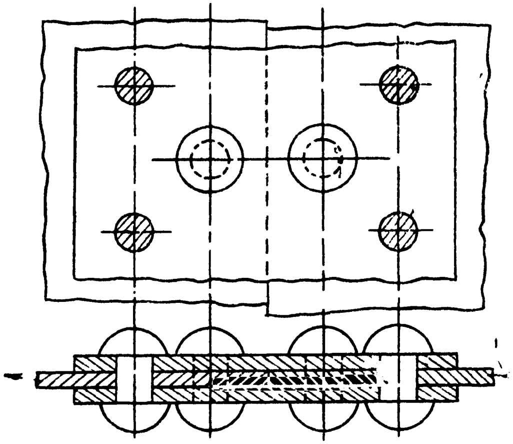


Fig. 174(*d*)

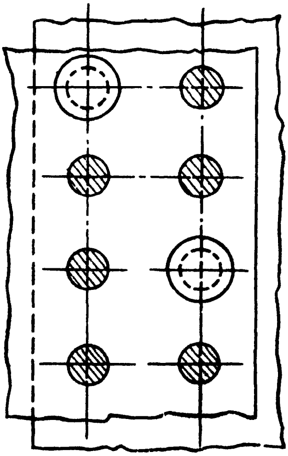


Fig. 174(*e*)

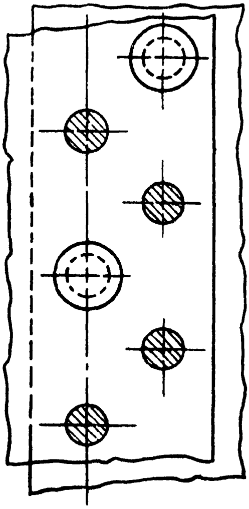


Fig. 174(*f*)

The distance, centre to centre, of rivets in a row is known as the pitch of rivets and is usually denoted by the letter *p*. Since we are concerned with the strength of a joint, we shall look upon the pitch as the length of plate in charge of a rivet (or rivets, as the case may be) which is responsible for its safety.

Single and double shear:

When the jointed plates are in tension, a rivet section is likely to be sheared off and will have to set up shear stresses against the pull on the plates. In fig. 174(*a*) and 174(*b*) and the single cover butt joint in fig. 174(*c*), the pull has to be resisted by *one* section of a rivet. The rivet, therefore, is in single shear; while in the double cover butt joint in fig. 174(*c*) and in fig. 174(*d*), the pull will be resisted by *two* section of each rivet. The rivets, are, therefore, in double shear.

Thickness of cover plates:

If t is the thickness of plates to be connected by a butt joint, it is evident that the thickness of cover plate should be at least t for a single cover and at least $\frac{t}{2}$ for two covers. To ensure against any possibility of failure of cover plates, the thickness provided is invariably more, being of the order of $1\frac{1}{8}t$ for single cover and $\frac{5}{8}t$ for double cover.

3. Possible ways of failure of a riveted joint: Let us consider a single riveted lap joint in a plate. Failure will take place either by the plate giving way or by the rivets failing. We need consider only a pitch-length of plate, as shown in fig. 175(*a*) since the rivet is responsible for that length of plate.

The plate can fail by tearing off across the pitch length as shown in fig. 175(*c*) or it can tear off at the edge, as shown in fig. 175(*d*), if the distance of the edge of the plate from the rivet hole is not sufficient. Such a failure of the joint is inexcusable and can be avoided by making the distance of the edge of the plate from the centre of the rivet not less than $1\frac{1}{2}d$, where d is the diameter of the rivet.

The rivet can fail in two ways. It will either shear off across its section under the pull on the jointed plate, or it will

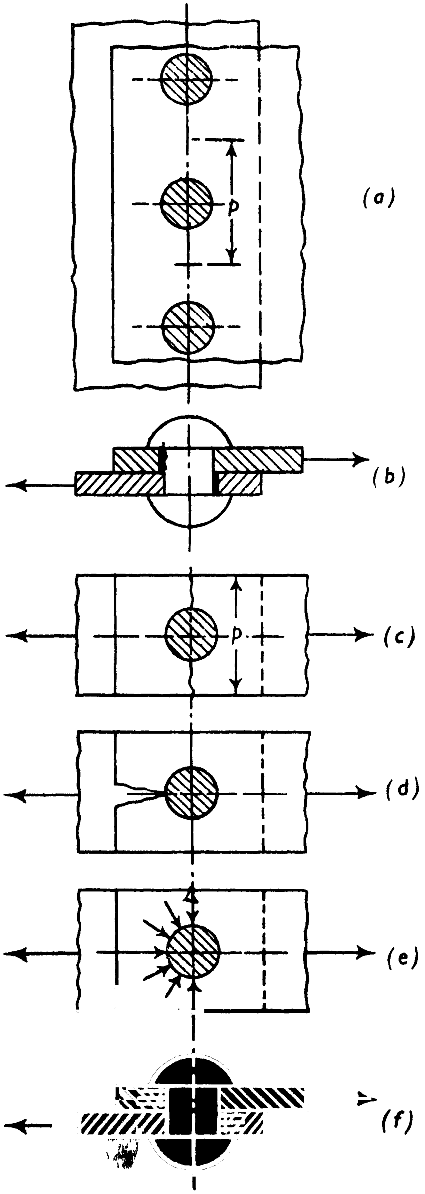


Fig. 175

get crushed. The pull on the plates will exert radial pressure as shown in fig. 175(*e*), across the thickness of plate, on the upper part of the rivet in the top plate and the lower part of the rivet in the bottom plate, as shown in fig. 175(*f*) and also by the thickened lines in fig. 175(*b*). This pressure will be compressive and if the rivet is not able to resist it, it will fail by crushing.

4. Strength of a riveted joint: A riveted joint will, therefore, fail either (i) by the plate tearing across the pitch length, or (ii) by the rivet shearing off or getting crushed.

In a given joint, it is possible to estimate the pull on the plate at which the joint will fail and also the manner of its failure, if the strength of the plate and rivet material is known. For instance, if f_t is the ultimate tensile strength per sq. inch for the plate-material and f_s and f_b the ultimate shearing and crushing stresses per sq. inch, respectively, for the rivet-material, we can calculate the pulls required on the plate to (i) tear the plate, (ii) shear the rivet and (iii) crush the rivet.

Per pitch length p:

(i) The pull required to tear the plate,

$P_t = f_t \times$ resisting section

$= f_t\ (p - d)\ t$, where d is the diameter of the rivet-hole and t is the thickness of plate.

(ii) The pull required to shear the rivet,

$P_s = f_s \times \frac{\pi}{4} d^2$, if the rivet is in single shear.

or $= f_s \times 2 \times \frac{\pi}{4} d^2$, if the rivet is in double shear.

(iii) The pull required to crush the rivet,

$P_b =$ intensity of radial pressure $\times$ projected area

$= f_b \times d \times t.$

Obviously, failure will occur at the least of these three pulls, P_t, P_s or P_b, because a higher value of the pull will never be reached, the joint having already failed, either by the tearing off of the plate, or the shearing or crushing of the rivet. The strength of the joint is, thus, the least of P_t, P_s and P_b. The efficiency of the joint will be the ratio of the strength of the joint and the strength of an unjointed plate. The pull required to tear a solid plate will be $P = f_t \times p \times t$. The efficiency of the joint will be given by,

$$\eta = \frac{\text{least of } P_t,\ P_s \text{ and } P_b}{P}.$$

If a joint is double riveted as in fig. 174(*b*) or fig. 174(*d*), two rivets are responsible for the safety of the pitch length p of the plate. In this case,

Per pitch length:

$P_t = (p \quad d)\ t\ f_t$, failure taking place across *either* row of rivets.

$P_s = 2 \times \frac{\pi}{4} d^2 \times f_s$ for rivets in single shear,

or $\quad = 2 \times 2 \times \frac{\pi}{4} d^2 \times f_s$ for rivets in double shear and

$P_b = 2 \times d.t.f_b$.

Similarly, if there are n rows of rivets with pitch p, n rivets are in charge of the pitch length of the plate. And,

Per pitch length:

$P_t = (p \quad d)\ t\ f_t$, failure taking place across *any* row of rivets.

$P_s = n \times \frac{\pi}{4} d^2 \times f_s$, for rivets in single shear,

or $\quad = n \times 2 \times \frac{\pi}{4} d^2 \times f_s$, for rivets in double shear and

$P_b = n \times d.t.f_b$.

Example 1.

A double riveted double-cover butt joint in plates $\frac{5}{8}''$ *thick is made with* $1''$ *rivets, at* $4''$ *pitch. If* f_t *for the plates is 28 tons/in*^{2}*., and* f_s *and* f_b *for rivets are 20 and 40 tons/in*^{2}*., respectively, calculate the pull per pitch length at which the joint will fail and also its efficiency.*

Per pitch length of 4″,

$P_t = (p - d)\, t\, f_t = (4 - 1) \times \frac{5}{8} \times 28 = 52{\cdot}5$ tons.

Since two rivets are in charge of the pitch length,

$$P_s = 2 \times 2 \times \frac{\pi}{4} d^2 \times f_s = 4 \times {\cdot}7854 \times 20 = 62{\cdot}83 \text{ tons.}$$

and $P_b = 2 \times d \times t \times f_b = 2 \times 1 \times \frac{5}{8} \times 40 = 50$ tons.

The joint will fail at a pull of 50 tons on the plates, the rivets giving way by being crushed.

The strength of the solid plate,

$P = p.t.f_t = 4 \times \frac{5}{8} \times 28 = 70$ tons.

$$\text{The efficiency of the joint} = \frac{\text{strength of joint}}{\text{strength of solid plate}}$$

$$\therefore \qquad \eta = \frac{50}{70} \text{ or } 71{\cdot}4\%.$$

Example 2.

Two single-riveted lap joints in $\frac{1}{2}''$ *plates are made as under:*

(*i*) *with* $\frac{7}{8}''$ *rivets at* $2\frac{1}{2}''$ *pitch,*

(*ii*) *with* $1\frac{1}{8}''$ *rivets at* $3\frac{1}{2}''$ *pitch.*

State which joint is stronger; given, the permissible stresses as 8 tons/in^2*., in tension for the plates and 6 tons/in*^{2}*., and 10 tons/in*^{2}*., in shear and bearing respectively for rivets.*

Taking the first joint,

Per pitch length of $2\frac{1}{2}''$,

$P_t = (p - d)\, t f_t = (2\frac{1}{2} - \frac{7}{8}) \times \frac{1}{2} \times 8 = 6{\cdot}5$ tons.

$$P_s = \frac{\pi}{4} d^2 \times f_s = \frac{\pi}{4} \left(\tfrac{7}{8}\right)^2 \times 6 = 3{\cdot}6 \text{ tons.}$$

$P_b = dt\, f_b = \frac{7}{8} \times \frac{1}{2} \times 10 = 4{\cdot}375$ tons.

The maximum permissible pull on this joint is therefore 3·6 tons. A solid plate would have taken a pull of $P = p.t.f_t = 2\frac{1}{2} \times \frac{1}{2} \times 8 = 10$ tons.

The efficiency of this joint is,

$$\eta_1 = \frac{3{\cdot}6}{10} \text{ or } 36\%.$$

Taking the second joint,

Per pitch length of $3\frac{1}{2}''$,

$$P_t = (p - d)\ tf_t = (3\tfrac{1}{2} - 1\tfrac{1}{8}) \times \tfrac{1}{2} \times 8 = 9{\cdot}5 \text{ tons.}$$

$$P_s = \frac{\pi}{4} d^2 \times f_s = \frac{\pi}{4}\ (1\tfrac{1}{8})^2 \times 6 = 6 \text{ tons.}$$

$$P_b = dt\ f_b = 1\tfrac{1}{8} \times \tfrac{1}{2} \times 10 = 5{\cdot}625 \text{ tons.}$$

The maximum pull this joint will safely take is, thus, 5·625 tons. The safe pull on a solid plate of this pitch length is $P = p.t.\ f_t = 3\frac{1}{2} \times \frac{1}{2} \times 8 = 14$ tons.

The efficiency of this joint is given by,

$$\eta_2 = \frac{5{\cdot}625}{14} = 40{\cdot}2\%.$$

The second joint, with its higher efficiency, is stronger.

5. Design of a riveted joint: The thickness of the plates to be jointed is usually designed so that the tensile stress at the section or sections through the rivet holes does not exceed the working stress for the material. We have, therefore, only to select a suitable diameter of rivets and then calculate their spacing or pitch.

If *d* is the diameter of rivets, let *p* be the pitch i.e. the length of the plate to be entrusted to the rivet or rivets in charge.

The pull to be taken by the plate, per pitch length, is $P_t = (p - d)\ t.\ f_t$ where f_t is the permissible tensile stress in the plate. Under this pull, the permissible stresses in the rivet, for shearing and bearing must not be exceeded. If f_s and f_b respectively are the permissible values for these stresses, the pull in the plate at which f_s will be reached in the rivet is given by,

$$P_s = \frac{\pi}{4} d^2 \times f_s, \text{ if the rivet is in single shear.}$$

or $$= 2 \times \frac{\pi}{4} d^2 \times f_s, \text{ [illegible] the rivet is in double shear.}$$

Similarly, the pull in the plates at which f_b will be reached in the rivets is $P_b = d.t.\ f_b$ for a single riveted joint.

In order that neither f_s nor f_b in the rivet is exceeded, the permissible pull on the plates will be P_s or P_b, whichever is less.

The pitch of rivets will therefore be determined on the basis that the pull $P_t = (p \quad d)\ t . f_t$, coming on the pitch length of the plate must not exceed the strength of the rivet or rivets in charge, i.e.,

$P_t \leqq$ the strength of the rivet or rivets in charge
or $(p \quad d)\ t f_t \leqq$,, ,, ,, ,, ,,
$\leqq P_s$ or P_b, whichever is less, from which, the pitch p can be obtained.

Of the two elements of the joint, viz., the plate and the rivet, the rivet will thus be the stronger, so that if the plate does not fail, the rivet certainly will not fail. The strength of the joint is, evidently, P_t. The efficiency of the joint, if it is designed in this manner, is given by,

$$\eta = \frac{P_t}{P} = \frac{(p \quad d)tf_t}{p.t.f_t} = \frac{p \quad d}{p}.$$

For the size of rivets to be used for plates of a given thickness t, a suitable diameter may be selected. As a guide, Unwin's formula $d = 1{\cdot}2\sqrt{t}$ may be used, where d is the diameter of rivets and t the thickness of plates in inch units.

Example 3.

Calculate the pitch and diameter of a single-riveted lap-joint in plates $\frac{3}{8}''$ thick, if the permissible stresses are:

$f_t = 5\frac{1}{2}$ tons/in^2., for plates; $f_s = 5$ tons/in^2.
$f_b = 12$ tons/in^2., for rivets.

Using Unwin's formula,

$d = 1{\cdot}2\sqrt{t} = 1{\cdot}2\sqrt{\frac{3}{8}} = {\cdot}735''$ say, $\frac{3}{4}''$.

The strength of one $\frac{3}{4}''$ diameter rivet in single shear is,

$$P_s = \frac{\pi}{4}\, d^2 \times f_s = \frac{\pi}{4}\left(\tfrac{3}{4}\right)^2 \times 5 = 2{\cdot}2 \text{ tons.}$$

The strength of one $\frac{3}{4}''$ diameter rivet in bearing is,

$$P_b = dtf_b = \tfrac{3}{4} \times \tfrac{3}{8} \times 12 = 3{\cdot}375 \text{ tons.}$$

The maximum permissible pull on the plates is therefore, the strength of one rivet i.e. $P_s = 2{\cdot}2$ tons. If p is the pitch of rivets,

$$P_t = (p - d)\ tf_t = (p - \tfrac{3}{4}) \times \tfrac{3}{8} \times 5{\cdot}5 = \tfrac{33}{16}\,(p - \tfrac{3}{4}) \text{ tons.}$$

This must not exceed the strength of the rivet in charge.

$$\therefore\quad \tfrac{33}{16}\,(p - \tfrac{3}{4}) \leqq 2{\cdot}2$$

$$\therefore\quad p \leqq {\cdot}75 + 1{\cdot}07$$

$$\leqq 1{\cdot}82.$$

Make $p = 1\tfrac{3}{4}''$. The efficiency of the joint will be,

$$\eta = \frac{p - d}{p} = \frac{1\tfrac{3}{4} - \tfrac{3}{4}}{1\tfrac{3}{4}} = \tfrac{4}{7} = 57{\cdot}2\%.$$

Example 4.

Calculate the pitch of $\tfrac{7}{8}''$ diameter rivets necessary for a double-riveted, double-cover butt joint in plates $\tfrac{1}{2}''$ thick, if f_t for plates = 6 tons/in^2.; f_s for rivets = 5 tons/in^2. and f_b = 10 tons/in^2.

If p is the pitch of rivets, the pull to be taken is,

$$P_t = (p - d)\ tf_t = (p - \tfrac{7}{8}) \times \tfrac{1}{2} \times 6 = 3\ (p - \tfrac{7}{8}) \text{ tons.}$$

The strength of one rivet, in double shear, is,

$$P_s = 2 \times \frac{\pi}{4}d^2 \times f_s = 2 \times \frac{\pi}{4} \times (\tfrac{7}{8})^2 \times 5 = 6 \text{ tons and in bearing}$$

$$P_b = dt\,f_b = \tfrac{7}{8} \times \tfrac{1}{2} \times 10 = 4{\cdot}375 \text{ tons.}$$

The strength of one rivet is 4·375 tons only. Since the joint is double riveted, two rivets are responsible for the safety of the pitch length of plate.

$$\therefore\quad P_t \leqq 2 \times 4{\cdot}375$$

$$\therefore\quad 3\ (p - \tfrac{7}{8}) \leqq 8{\cdot}75$$

$$\therefore\quad p \leqq {\cdot}875 + 2{\cdot}917$$

$$\leqq 3{\cdot}792''.$$

Therefore make $p = 3\tfrac{1}{2}''$.

The thickness of cover plates will be $\tfrac{5}{8}t$ or $\tfrac{5}{16}''$ in this case.

The efficiency of the joint $= \dfrac{p - d}{p} = \dfrac{3\tfrac{1}{2} - \tfrac{7}{8}}{3\tfrac{1}{2}} = \tfrac{21}{28}$ i.e. 75%.

Some text-books suggest the design of a riveted joint such that it is equally strong in tension of plates and shearing and bearing of rivets. If p is the pitch length, the pull to be taken is $P_t = (p - d)\ tf_t$. If, at this pull, the rivet must reach the permissible shear stress f_s as well as the permissible bearing stress f_b, then evidently it must have a particular diameter such that $P_s = P_b$. The pitch will then be obtained by equating P_t to P_s or P_b. With this diameter of rivets and the pitch thus obtained, $P_t = P_s = P_b$ and the joint will be equally strong in tension of plates and shearing and bearing of rivets.

Equating P_s to P_b, we get, for a rivet in single shear,

$$\frac{\pi}{4}d^2 \times f_s = d.t.f_b$$

$$\therefore \qquad d = \frac{4f_b}{f_s} \times \frac{t}{\pi}.$$

It will be noticed that the expression for d is a function of π and it is impossible to adopt the diameter so obtained; the size has to be approximated to the nearest eighth of an inch. As soon as this is done, P_s is no longer equal to P_b and the strength of the rivet is the lesser of P_s or P_b as already explained.

The equations $P_t = P_s = P_b$,

i.e., $(p \quad d)\ tf_t = \frac{\pi}{4}d^2.\ f_s = dt.f_b$, to obtain the pitch and diameter of rivets, have little practical value.

In calculations for riveted joints, areas of circular sections ranging from $\frac{1}{4}''$ to $1\frac{1}{4}''$ diameter are necessary.

The following table will be found useful, for the purpose:

Diameter	Area	Diameter	Area
$\frac{1}{4}$	·049	$\frac{7}{8}$	·60
$\frac{3}{8}$	·11	1	·7854
$\frac{1}{2}$	·1963	$1\frac{1}{8}$	1·00
$\frac{5}{8}$	·30	$1\frac{1}{4}$	1·23
$\frac{3}{4}$	·44		

6. Riveted joints in boiler-shells: In designing the thickness of the boiler shell from the formula $t = \frac{pd}{2f.\eta}$, the efficiency η of the longitudinal joint is required. But the efficiency of the joint cannot be known until the thickness of the plates to be jointed is known. It is therefore usual to assume a suitable value for the efficiency of the longitudinal joint — about 70 to 75% — and design the thickness of plate required for the shell. Next, the longitudinal joint is designed as required and its efficiency calculated. If the efficiency of the actual joint is not less than the assumed value, the design will be satisfactory.

Example 5.

A boiler shell, 6 feet diameter, has to withstand a steam pressure of 180 lb/in². Design the thickness of the shell for a permissible tensile stress in the plate of 6 tons/in²., assuming the efficiency of the longitudinal joint at 70%.

Calculate the pitch of rivets required for the joint, which is to be double-cover, double-riveted, using 1" rivets. For the rivets, the permissible stresses are $f_s = 5$ tons/in²., and $f_b = 10$ tons/in². Check the efficiency of the joint.

For the necessary thickness of plate,

$$t = \frac{pd}{2f.\eta} = \frac{180 \times 72}{2 \times 6 \times 2240 \times \cdot 70} = \cdot 6889'', \text{ say, } \tfrac{3}{4}''.$$

If p is the pitch of rivets, the pull to be taken per pitch length is,

$$P_t = (p - d)\, t\, f_t = (p - 1) \times \tfrac{3}{4} \times 6 = \tfrac{9}{2}\,(p - 1) \text{ tons.}$$

The strength of a rivet in double shear is,

$$P_s = 2 \times \cdot 7854 \times 5 = 7\cdot 854 \text{ tons and in bearing,}$$

$$P_b = 1 \times \tfrac{3}{4} \times 10 = 7\cdot 5 \text{ tons.}$$

The safe pull on the plates for one rivet is thus 7·5 tons. Since two rivets are in charge of the pitch length,

$$P_t = \tfrac{9}{2}\,(p - 1) \leqq 2 \times 7\cdot 5$$

$$\therefore\ (p - 1) \leqq 3\cdot 33$$

$$\therefore\ p \leqq 4\cdot 33.$$

Make $p = 4''$. The efficiency of the actual joint will be $\frac{p-d}{p} = \frac{4-1}{4} = 75\%$ against the assumed value of 70%.

7. Riveted joints in structural steel work: In structural steel work, the problem is slightly different. The pull or push to be transmitted by a member of a roof truss or bridge truss is known. The problem is to design the cross-section of the member and to provide a sufficient number of rivets of a given diameter to transmit the pull or push. If *P* is the action to be transmitted, the number of rivets necessary

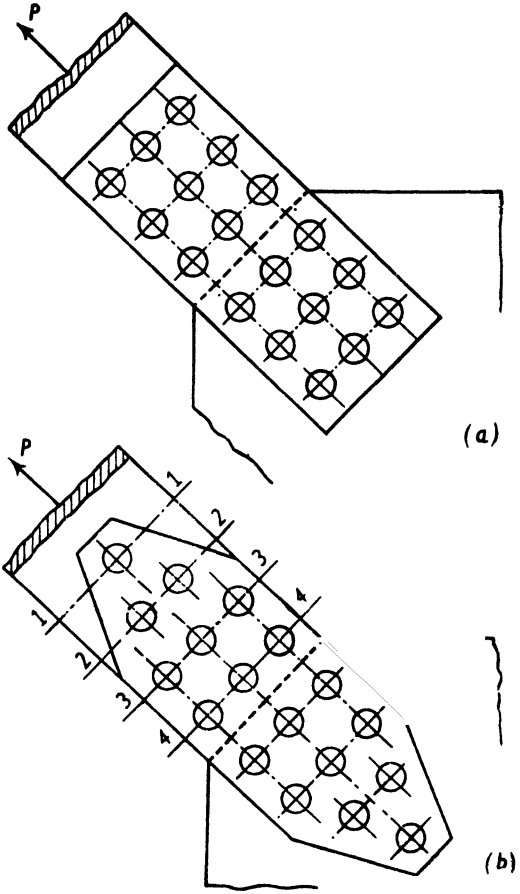


Fig. 176

will be given by $n = \dfrac{P}{\text{strength of one rivet}}$, the strength of the rivet being P_s or P_b for the rivet, whichever is less. If this or a larger number of rivets is used, the joint is safe against failure of rivets.

As for the sectional area required for the member, if it is in compression, Rankine's or any other suitable empirical formula may be used. For a tension member, the section will be obtained from the relation $P = f.A$, where f is the permissible tensile stress and A is the net area of the weakest section, deducting rivet holes, so that the weakest section is designed to resist the pull P at the safe working stress.

In the case of tension members of a roof truss, the number of rivets required for the joint is small and the section is usually weakened by one rivet hole only. In a bridge truss, however, the number of rivets required is large and the section will depend upon how these rivets are arranged. For instance, suppose a bridge diagonal is a flat, of thickness t, needing 9 rivets to connect it by a butt joint to the gusset plate, we may arrange the rivets in three rows of three each as in fig. 176(a) or, as shown in fig. 176(b).

The rivets have thus been provided for. Now for the width of flat that is necessary to transmit a given pull P, it must be such that at the weakest section, the tensile stress does not exceed the working stress f. For the joint in 176(a), the section is weakened by three rivets holes of diameter d. The width b of the flat must therefore be such that,

$$P = f(b \quad 3d).t$$

or $$b = (\frac{P}{ft} + 3d)$$

For the joint in fig. 176(b), assuming the section 11, through the first rivet hole to be the weakest, the width of flat will be given by,

$$P = f(b - d).t$$

or $b = (\dfrac{P}{ft} + d)$, which is less by $2d$ than if the first design is adopted. In a long bridge diagonal, a reduction in the required width for the member, by a suitable grouping of rivets, is a consideration. We shall, therefore, adopt the second grouping—known as diamond riveting, shown in fig. 176(*b*)—in preference to the chain-riveting shown in fig. 176(*a*).

In diamond riveting, section 11, which is weakened by one rivet hole only, is usually the weakest, sections 22, 33, 44, [see fig. 176(*b*)] successively getting stronger. If f_t is the limit of tensile stress for the plate, the pull required to tear the plate at section 11 is given by,

$$P_1 = f_t\ (b \cdot\ d)t.$$

To fail at section 22 by tearing of the plate, the rivet in front must first give way. The pull required for the failure of the plate at section 22 is given by,

$$P_2 = f_t\ (b - 2d)t + \text{strength of one rivet in front.}$$

Similarly, for failure at section 33 by tearing of the plate, the three rivets in front must give way, and therefore the pull required for tearing at section 33 is given by,

$$P_3 = f_t(b - 3d)\ t + \text{strength of three rivets in front.}$$

Similarly,

$P_4 = f_t(b - 3d)\ t +$ strength of six rivets in front, for failure at section 44.

Of the pulls P_1, P_2, P_3 and P_4 usually P_1 is the least, which means that the weakest section is 11. The efficiency of such a joint is the greatest, being,

$$\eta = \frac{P_1}{P} = \frac{(b-d)\,t f_t}{b.t f_t} = \frac{b-d}{b}.$$

Such a joint thus provides the most economical arrangement of rivets.

Example 6.

A bridge diagonal, $\frac{1}{2}''$ thick, has to transmit a pull of 42 tons. It is to be connected to a $\frac{1}{2}''$ gusset plate by a double cover butt joint with $\frac{3}{4}''$ rivets. Calculate the number of rivets that will be necessary and the width of flat required if the permissible tensile stress in the flat is $7\frac{1}{2}$

tons/in²., and the shearing and bearing stresses in the rivets are not to exceed 6 tons/in²., and 12 tons/in²., respectively. Sketch the joint and calculate its efficiency. Calculate also the actual stresses induced in the flat and the rivets.

The strength of one rivet in double shear is,

$$P_s = 2 \times \cdot 44 \times 6 = 5{\cdot}28 \text{ tons.}$$

The strength of one rivet in bearing in ½″ plate is,

$$P_b = \tfrac{3}{4} \times \tfrac{1}{2} \times 12 = 4{\cdot}5 \text{ tons.}$$

One rivet can take a pull of 4·5 tons only.

The number of rivets required will be,

$$n = \frac{42}{4{\cdot}5} = 9{\cdot}33, \text{ say, } 10.$$

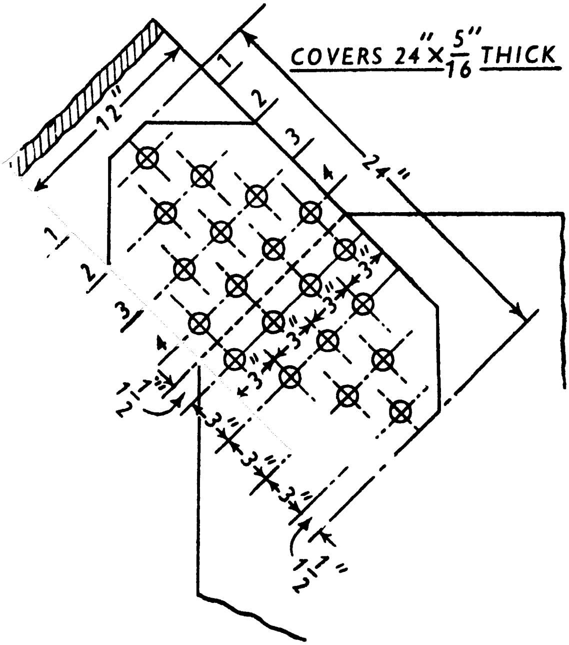


Fig. 177

Assuming the section to be weakened by one rivet hole only,

$$P_t = (b - d)t.f_t = (b - \tfrac{3}{4}) \times \tfrac{1}{2} \times 7{\cdot}5 = 42 \text{ tons.}$$

$$\therefore\quad b = \cdot 75 + 11{\cdot}2 = 11{\cdot}95'', \text{ say, } 12''.$$

The efficiency of the joint $= \frac{b-d}{b} = \frac{12-\frac{3}{4}}{12} = \frac{11{\cdot}25}{12}$

$= 93{\cdot}75\%.$

Actual stresses induced in the joint:

For rivets, since 10 rivets take up the pull of 42 tons, each rivet gets 4·2 tons.

The actual shear stress induced in each rivet

$$= \frac{4{\cdot}2}{2 \times {\cdot}44} = 4{\cdot}77 \text{ tons/in}^2.$$

The actual bearing stress per rivet

$$= \frac{4{\cdot}2}{\frac{3}{4} \times \frac{1}{2}} = 11{\cdot}2 \text{ tons/in}^2.$$

For the flat, if f_1 is the tensile stress at section 11,

$$f_1 \ (12 - \tfrac{3}{4}) \times \tfrac{1}{2} = 42$$

$$\therefore \ f_1 = 7{\cdot}466 \text{ tons/in}^2.$$

At section 22, the tensile stress f_2 in the flat will be given by,

$$P = 42 = f_2 \ (b \quad 2d) \ t + \frac{P}{n}.$$

$$\therefore \ f_2 \ (12 - 2 \times \tfrac{3}{4}) \times \tfrac{1}{2} + \frac{42}{10} = 42$$

$$\therefore \ 5{\cdot}25 f_2 = 37{\cdot}8$$

and $\quad f_2 = \frac{37{\cdot}8}{5{\cdot}25} = 7{\cdot}2 \text{ tons/in}^2.$

At section 33,

$$P = 42 = f_3 \ (b - 3d).t + \frac{3P}{n}$$

$$\therefore \ f_3 \ (12 - 3 \times \tfrac{3}{4}) \times \tfrac{1}{2} + 3 \times 4{\cdot}2 = 42$$

$$\therefore \ 4{\cdot}875 f_3 = 29{\cdot}4$$

and $\quad f_3 = \frac{29{\cdot}4}{4{\cdot}875} = 6{\cdot}032 \text{ tons/in}^2.$

At section 44,

$$P = 42 = f_4 \ (b \quad 4d) \times t + \frac{6P}{n}$$

$$\therefore\quad f_4\,(12-4\times\tfrac{3}{4})\times\tfrac{1}{2}=42-\frac{6\times 42}{10}$$

$$\therefore\quad 4{\cdot}5f_4=16{\cdot}8$$

$$\text{and}\quad f_4=\frac{16{\cdot}8}{4{\cdot}5}=3{\cdot}734\ \text{tons/in}^2.$$

It is thus seen that sections 22, 33 and 44 are successively stronger.

Example 7.

A tie-bar in a bridge consists of a flat 9″ wide and $\frac{3}{4}$″ thick and is connected to a gusset plate of the same thickness by a double cover butt joint. Design an economical joint if the permissible stresses are 8 tons/in^2. in the plate and 5 tons/in^2., for shear and 10 tons/in^2., for bearing in rivets. Use $\frac{7}{8}$″ rivets.

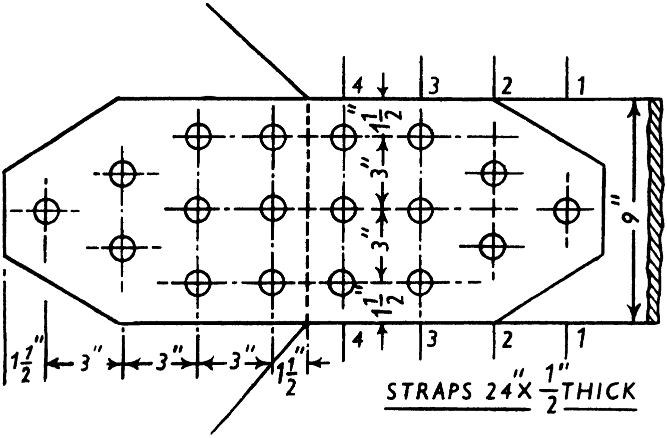


FIG. 178

Using the diamond grouping of rivets, the plate will be weakened by one rivet hole only. The maximum pull that can be transmitted at a tensile stress of 8 tons/in^2., in the flat will be,

$$P=(b-d)\ t.f_t=(9-\tfrac{7}{8})\times\tfrac{3}{4}\times 8=48{\cdot}75\ \text{tons}.$$

The strength of one $\frac{7}{8}$″ rivet in double shear is,

$P_s = 2\times{\cdot}6\times 5 = 6$ tons; and in bearing.

$$P_b=\tfrac{7}{8}\times\tfrac{3}{4}\times 10=6{\cdot}56\ \text{tons}.$$

The number of rivets necessary to transmit the pull of 48·75 tons will be,

$$n = \frac{48{\cdot}75}{6} = 8{\cdot}13, \text{ say, } 9.$$

The efficiency of the joint is,

$$\eta = \frac{b-d}{b} = \frac{9-\frac{7}{8}}{9} = \frac{65}{72} = 90{\cdot}3\%.$$

In the previous question we worked out the actual stresses set up by sections 11, 22, 33 and 44 of the flat against a given pull. We may also estimate the comparative strengths of the sections as under:

At the permissible stress of 8 tons/in^2., in tension, the safe pull P_1 on section 11 will be given by,

$$P_1 = f\ (b\ -d) \times t = 8\ (9\quad \tfrac{7}{8}) \times \tfrac{3}{4} = 48{\cdot}75 \text{ tons.}$$

At the same stress, the pull P_2 for section 22 will be,

$$P_2 = f\ (b\quad 2d) \times t + \text{strength of one rivet in front.}$$
$$= 8\ (9 - 2 \times \tfrac{7}{8}) \times \tfrac{3}{4} + 6 = 43{\cdot}5 + 6 = 49{\cdot}5 \text{ tons.}$$

Similarly for section 33,

$$P_3 = f\ (b\quad 3d) \times t + \text{strength of three rivets in front.}$$
$$= 8\ (9\ -3 \times \tfrac{7}{8}) \times \tfrac{3}{4} + 3 \times 6$$
$$= 38{\cdot}25 + 18 = 56{\cdot}25 \text{ tons.}$$

And for section 44,

$$P_4 = f\ (b\quad 3d) \times t + \text{strength of six rivets in front}$$
$$= 8\ (9\quad 3 \times \tfrac{7}{8}) \times \tfrac{3}{4} + 6 \times 6$$
$$= 38{\cdot}25 + 36 = 74{\cdot}25 \text{ tons.}$$

The sections get progressively stronger. The maximum permissible pull which the flat will safely transmit is $P_1 = 48{\cdot}75$ tons at section 11, which is the weakest.

Example 8.

Find the number of rivets necessary for the gusset plate of a steel stanchion transmitting a load of 150 tons. Thickness of plate $\frac{3}{4}''$; diameter of rivets $1''$; permissible stress in rivets, $f_s = 6$ tons/in^2., and $f_b = 12$ tons/in^2.

The rivets are in single shear.

$P_s = \cdot 7854 \times 6 = 4\cdot 71$ tons.

$P_b = 1 \times \frac{3}{4} \times 12 = 9$ tons.

The number of rivets required is,

$$n = \frac{150}{4\cdot 71} = 31\cdot 85, \text{ say } 32.$$

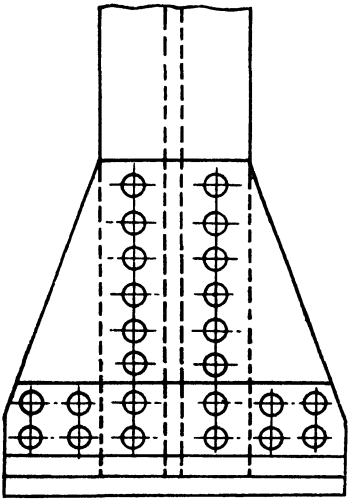


FIG. 179

The arrangement of rivets is shown in fig. 179. There are two gusset plates, one on either side of the flanges connecting the stanchion with the base plate by means of rivets and angle-cleats.

8. Pitch of rivets in built-up girders: In a plate girder in which the section is built-up of plates connected with angles, it is necessary to calculate the pitch of rivets, particularly at the section where the shear force is maximum—usually at the supports.

Consider a section at which the shear force is *F*. We have seen, in the study of beams, that a vertical shear stress across a beam section is accompanied by a horizontal shear of the same intensity. Let *p* be the pitch of rivet connecting the angles to the web plate. These rivets are in double shear. If *R* is the strength of one rivet (which is P_s or P_b for the rivet, whichever is less), the length of web plate to be placed in its charge must

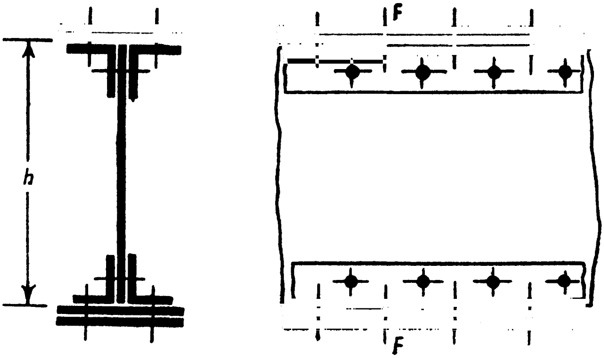


FIG. 180

be such that the shear effect across it does not exceed R. Assuming for simplicity that the web takes the whole shear, the intensity of vertical and horizontal shear stress at the section is given by,

$q = \frac{F}{t.h}$, where t is the thickness and h the depth over angles of the web plate. If p is the pitch of rivets, the shear effect on a pitch length $= q \times p \times t = \frac{F}{t.h} \times p \times t = \frac{F.p}{h}$

$$\therefore \quad \frac{F.p}{h} \leqq R$$

$$\text{or} \quad p \leqq \frac{R.h}{F}$$

The pitch of rivets connecting the angles and the flange plate will be the same. These rivets are in single shear; but, for each rivet connecting the angles to the web plate, there are two to connect the angles to the flange plates.

The pitch varies inversely as the shear force at the section, being minimum where the shear force is a maximum. It is sufficient to calculate the minimum pitch of rivets required at the supports and to increase it judiciously at sections away from the supports.

Example 9.

A plate girder, 50 feet span, depth of web 4 feet, is loaded with a uniformly distributed load of 3 tons per foot run. The web, which is $\frac{1}{2}''$ *thick, is connected to the flange plates by* $4'' \times 4'' \times \frac{1}{2}''$ *angles. Find the pitch of* $\frac{7}{8}''$ *rivets required to connect the angles to the web plate near the supports. The permissible stress for rivets are* $f_s = 6$ *tons/in².* *and* $f_b = 12$ *tons/in².*

For one rivet in double shear,

$$P_s = 2 \times \cdot 6 \times 6 = 7 \cdot 2 \text{ tons.}$$

In bearing,

$$P_b = \tfrac{7}{8} \times \tfrac{1}{2} \times 12 = 5 \cdot 25 \text{ tons.}$$

$$\therefore \quad R = 5 \cdot 25 \text{ tons.}$$

The maximum shear at supports is,

$$F = \frac{50 \times 3}{2} = 75 \text{ tons.}$$

The minimum pitch required is,

$$p = \frac{R.h}{F} = \frac{5 \cdot 25 \times 48}{75} = 3 \cdot 36'', \text{ say, } 3''.$$

If the pitch comes too small, two rows of rivets at double the calculated pitch may be provided. This will be near enough.

9. Eccentric loading on rivets: If an eccentric load P comes on a group of rivets, the eccentricity of its line of action from the centre of gravity of the rivet system being e, it is equivalent to an axial force P through the c.g. accompanied by a bending moment $M = P.e.$ If the number of rivets is n, each rivet will have to resist a direct force $\frac{P}{n}$ and a moment coming to its share from the bending moment M. If (x, y) are the coordinates of the centre of any rivet, referred to the axes of symmetry through G, the centre of gravity of the rivet group (see fig. 182), it is assumed that the effect of the bending moment on the rivet is a tangential force F at right angles to the radius vector and proportional to it.

$\therefore \quad F \propto r$

or $\quad F = k.r.$

The moment transmitted to the rivet $= F.r = k.r^2$. The sum of all the moments transmitted by the rivets

$$= \Sigma k r^2 = M$$

$$\therefore \quad k = \frac{M}{\Sigma r^2}$$

and $F = kr = \dfrac{M.r}{\Sigma r^2}$.

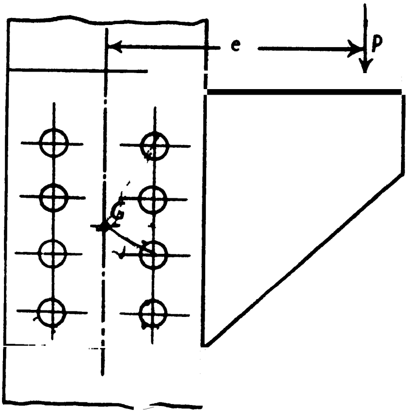


FIG. 181

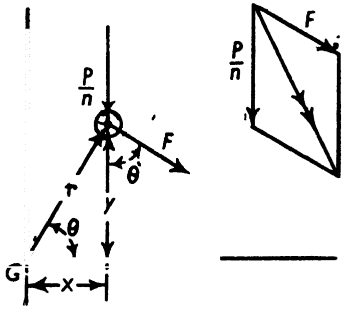


FIG. 182

Each rivet has to resist the resultant of the direct force $\frac{P}{n}$ and the tangential force F on it. This resultant may be obtained graphically as shown in fig. 182 or calculated. The vertical component V of the resultant $= (\frac{P}{n} + F\cos\theta)$ and the horizontal component $H = F\sin\theta$. The resultant $= \sqrt{V^2 + H^2}$, its inclination φ to the horizontal being given by, $\tan\varphi = \frac{V}{H}$.

Example 10.

A line shaft transmits a load of 2·4 tons across a bracket riveted to a stanchion as shown in fig. 181. The load eccentricity is 18″. If the number of $\frac{7}{8}$″ *rivets is 8, arranged 4″ centre to centre, vertically and horizontally, calculate the maximum shear intensity induced in any rivet.*

$M = P.e = 2{\cdot}4 \times 18 = 43{\cdot}2$ ton-inches.

The direct load per rivet $= \frac{P}{n} = \frac{2{\cdot}4}{8} = {\cdot}3$ ton.

Referring to fig. 181, for the rivets,

$$\Sigma r^2 = \Sigma\ (x^2 + y^2) = 4(2^2 + 6^2) + 4\ (2^2 + 2^2)$$
$$= 160 + 32 = 192.$$

The maximum shear effect F due to the B.M. will come on the rivets at the extreme ends, being,

$$F = \frac{M\sqrt{40}}{192} = \frac{43{\cdot}2 \times 6{\cdot}325}{192} = 1{\cdot}423 \text{ tons}$$

$$\tan\theta = \tfrac{6}{2} = 3 \quad \therefore \quad \theta = 71^\circ\ 36'.$$
$$\sin\theta = {\cdot}9489$$
$$\cos\theta = {\cdot}3156$$

$$V = \frac{P}{n} + F\cos\theta = {\cdot}3 + 1{\cdot}423 \times {\cdot}3156$$
$$= {\cdot}3 + {\cdot}449 = {\cdot}749 \text{ tons.}$$

$$H = F\sin\theta = 1{\cdot}423 \times {\cdot}9489 = 1{\cdot}35 \text{ tons.}$$

Resultant force on the extreme rivets

$$= \sqrt{{\cdot}749^2 + 1{\cdot}35^2}$$
$$= \sqrt{{\cdot}561 + 1{\cdot}822} = \sqrt{2{\cdot}383}$$
$$= 1{\cdot}544 \text{ tons.}$$

If the rivet is in single shear, the shear intensity

$$= \frac{1{\cdot}544}{{\cdot}6} = 2{\cdot}57 \text{ tons/in}^2$$

EXAMPLES XI

(1) A double riveted lap joint in plates $\frac{3}{8}''$ thick is made with $\frac{3}{4}''$ rivets at $3''$ pitch. Estimate how the joint will fail and calculate its efficiency if the tearing strength of the plates is 32 tons/in^2. and the shearing and bearing strength of the rivets are 20 tons/in^2. and 40 tons/in^2. respectively.

(By shearing of rivets: 48·9%.)

(2) A double riveted double cover butt joint is used to connect plates $\frac{1}{2}''$ thick. Using Unwin's formula, select a suitable diameter of rivets and calculate the pitch if the permissible stresses are $6\frac{1}{2}$ tons/in^2. in tension for plates, 5 and 10 tons/in^2. in shear and bearing respectively for rivets. What is the efficiency of the joint?

($d = \frac{7}{8}''$; $p = 3\frac{1}{2}''$; $\eta = 75\%$.)

(3) $\frac{3}{4}''$ plates are jointed together by a treble-riveted butt joint with two covers. The pitch of rivets in the outer rows is to be twice the pitch in the other rows. Using $\frac{7}{8}''$ diameter rivets, calculate the necessary pitch if the permissible tensile stress in the plates is 8 tons/in^2. and the shearing and bearing stresses in the rivets are not to exceed 5 and 10 tons/in^2. respectively.

(Pitch of outer rows $5\frac{3}{4}''$.)

(4) Plates $\frac{1}{2}''$ thick are jointed by a double riveted lap joint with $\frac{3}{4}''$ rivets at $2\frac{1}{2}''$ pitch. If f_t for the plates is 30 tons/in^2. and f_s and f_b for rivets are 24 and 40 tons/in^2. respectively, calculate the efficiency of the joint and state how it will fail. If this joint is strengthened by a cover plate as shown in fig. 183, calculate the efficiency of the strengthened joint, if the pitch of $\frac{3}{4}''$ rivets in the outer rows is $5''$. State how the joint will fail now.

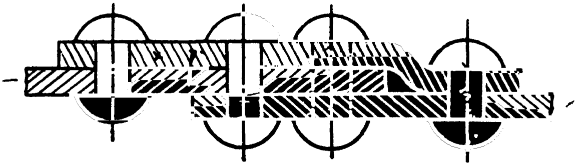


FIG. 183

(56·32%; rivets will shear off. 70·7%; ~~sh~~bearing of ~~rivets~~.

(5) A boiler shell, 5 feet in diameter, is made of $\frac{1}{2}''$ plates. The circumferential joint is a single riveted lap joint with $\frac{7}{8}''$ diameter rivets at a pitch of 2 inches. If the ultimate tensile stress in plates is 28 tons/in^2. and the ultimate shearing and bearing stresses for the rivets

are 20 and 40 tons/in^2. respectively, calculate the efficiency of the joint. When the steam pressure in the boiler is 150 lb/in^2., what is the factor of safety of the circumferential joint? (43%; 6.)

(6) In the above example, if the longitudinal joint is a double-riveted double cover butt joint, with $\frac{7}{8}''$ rivets at 3″ pitch, calculate its efficiency. Also, calculate the steam pressure that may be permitted in the boiler, using the stresses as above and a factor of safety of 4. (70·83%; 185 lb/in^2.)

(7) A tie bar 8″ wide and $\frac{3}{4}''$ thick has a lap joint as shown in fig. 184, made with $\frac{7}{8}''$ rivets. If ft for the flat is 30 tons/in^2. and fs and fb for the rivets are 20 and 40 tons/in^2. respectively, calculate the pull at which the joint will fail and state where it will give way. Also calculate the efficiency of the joint.

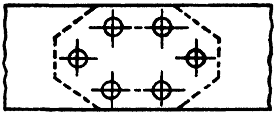


FIG. 184

(72 tons; shearing of all rivets; 40%.)

(8) At a joint in a roof truss, pulls of 9, 5 and 7 tons have to be transmitted axially across the tension members OA, OC and OD respectively and a thrust of 4 tons across the strut OB, as shown in fig. 185. Using $\frac{5}{8}''$ rivets and $\frac{1}{2}''$ thick flats for tension members, calculate their widths. Also calculate the number of rivets required to connect each member to the gusset plate if, the permissible stresses are 6 and 12 tons/in^2. in shear and bearing for rivets and 7 tons/in^2, in tension for flats. Sketch the joint.

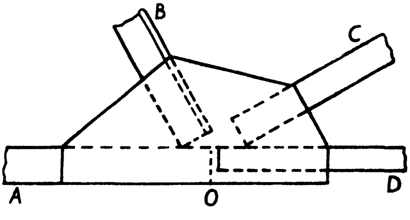


FIG. 185

$$\left\{\begin{array}{l} OA\colon b=3\tfrac{1}{4}'',\ n=5 \text{ rivets};\ OB\colon n=3;\ OC\colon \\ n=3,\ b=2\tfrac{1}{4}'';\ \ OD\colon b=2\tfrac{5}{8}'',\ n=4. \end{array}\right\}$$

(9) A tension diagonal in a framed girder is 12″ wide and $\frac{3}{4}''$ thick attached to a gusset plate of the same thickness with 1″ rivets. Design the joint so as to obtain the maximum strength if the permis-

sible stresses are 7½ tons/in². for tension in the flats, 5 and 10 tons/in². for shear and bearing in rivets. What is the efficiency of the joint? ($P = 61\frac{7}{8}$ tons; $n = 9$; $\eta = 91{\cdot}7\%$.)

(10) A bridge diagonal has to transmit a pull of 70 tons and is connected to a gusset plate by a double cover butt joint. The thickness of the flat is 1″, using ⅞″ rivets, find the number required for the joint with diamond riveting. Calculate also the width of flat and the actual shear and bearing stresses induced in the rivets, if the permissible stresses are 8 tons/in². for tension in flat, 6 tons/in². and 12 tons/in². for shear and bearing in rivets.

($b = 10''$; $n = 10$ rivets; 5·83 tons/in².; 8 tons/in².)

(11) Calculate the thickness of a flat tie-bar of a bridge truss, 10″ wide if it has to transmit a pull of 52 tons. It is to be attached to a gusset plate ¾″ thick by a double cover butt joint with ⅞″ rivets. Design an economical joint and state its efficiency if the permissible stresses are the same as in example 10. (¾″; 8 rivets; 91·25%.)

(12) A structural member is attached by a lap joint to a gusset plate by means of five rivets of ¾″ diameter, symmetrically arranged as shown in fig. 186. The member has to transmit an axial pull of 7½ tons and a clockwise moment of 15 ton-inches. Calculate the maximum intensity of shear stress induced in any rivet. (5·94 tons/in².)

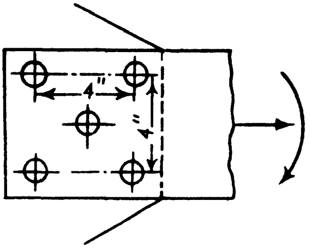


FIG. 186

(13) A group of 25 rivets, ⅞″ in diameter, is arranged in the form of a square at a pitch of 4″ horizontally and vertically. The group has to resist a vertical pull of 50 tons acting at the c.g. of the system and also a moment of 150 ton-inches. Estimate the maximum shear stress in any rivet. (4·75 tons/in².)

(14) A plate girder, 60 feet span, has a web 4 feet deep and ½″ thick connected to the flange plate by 4″ × 4″ × ½″ angles. If the equivalent uniformly distributed load over the girder is 3½ tons per foot run, calculate the minimum pitch for ¾″ rivets to connect the angles and the web. The permissible stresses for rivets are 6 tons/in². and 12 tons/in². for shear and bearing respectively.

(Two rows at 4″ pitch.)

CHAPTER XII

SHAFTS AND SPRINGS IN TORSION

1. Torsion: When a cylindrical shaft is subjected to equal and opposite couples at the ends, it will either be in equilibrium or it will rotate at a uniform speed. In either case, it is subjected to torsion and the stresses set up by every cross-section are shear stresses. At any point in the cross-section of a shaft, there is a state of simple shear, the two planes of shear, i.e., the planes across which the stress is wholly tangential, being (i) the cross-section itself and (ii) the plane through the point and the axis of the shaft.

2. Torsional stresses and strains: A shaft transmitting power rotates at a uniform speed. Whether it rotates at a uniform speed or is at rest, the stresses and strains due to equal and opposite couples at its ends will remain the same. It is therefore convenient to consider the shaft at rest in studying the stresses and strains.

Consider, then, a shaft of length l, radius R, with one end fixed.

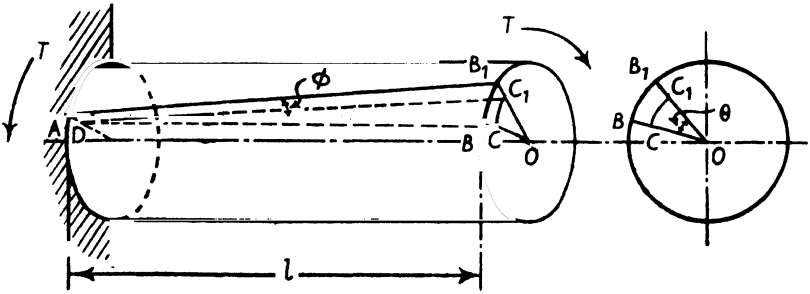


FIG. 187

Let a torque T be applied at the free end. The reaction at the fixed end will be a balancing couple of the same magnitude T.

Consider the deformation of the line AB on the surface of the shaft. As the cross-section distorts through an angle θ, AB distorts into AB_1, through an angle φ. If f_s is the intensity of shear induced at the surface and $\mathcal{N}$ the shear modulus of the material, we have,

$$\frac{f_s}{\mathcal{N}} = \varphi = \frac{BB_1}{AB} = \frac{OB.\theta}{AB} = \frac{R.\theta}{l} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

Consider, similarly, the deformation of an inside layer DC at a radius r from the central axis of the shaft, D being at the the fixed end. DC distorts to DC_1 through an angle φ_1. If q is the shear intensity at radius r, we have,

$$\frac{q}{\mathcal{N}} = \varphi_1 = \frac{CC_1}{DC} = \frac{OC.\theta}{DC} = \frac{r.\theta}{l} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

From (1) and (2), we get,

$$\frac{q}{r} = \frac{f_s}{R} = \frac{\mathcal{N}.\theta}{l}, \text{ which is a constant} \ldots\ldots\ldots\ldots\ldots\ldots(3)$$

We therefore deduce that the intensity of shear stress at any point in the cross-section of a circular shaft subjected to a pure twisting moment is proportional to its distance r from the axis of the shaft, varying uniformly from zero at the axis to a maximum f_s at the surface of the shaft. The central axis of the shaft is thus the neutral axis.

Moment of Resistance:

Consider an elementary ring of radius r and thickness δr in the cross-section of the shaft. The intensity of shear stress $q = \frac{f_s}{R}.r$. The resistance couple set up by the ring is equal to:

$$q \times 2\pi r dr \times r = \frac{f_s}{R} \times r^2 \times 2\pi r dr$$

$$= \frac{f_s}{R} \times r^2 \times \delta a,$$

where $\delta a = 2\pi r dr$ is the area of the elementary ring.

The total moment of resistance set up by the cross-section is given by,

$$T_r = \Sigma \frac{f_s}{R} \times r^2 \delta a = \frac{f_s}{R} \Sigma r^2.\delta a.$$

The expression $r^2\delta a$ is the moment of inertia of the elementary ring about the central axis and $\Sigma r^2 \delta a$ represents the moment of inertia of the entire section about the central axis. Using the letter J for the moment of inertia,

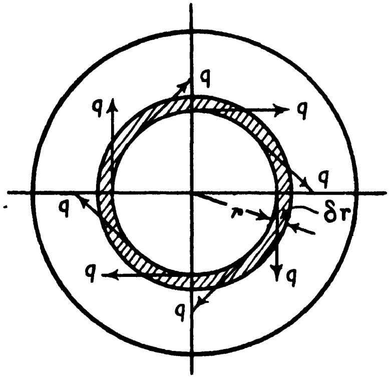


FIG. 188

we get,

$$T_r = f_s \times \frac{J}{R} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (4)$$

J is called the polar moment of inertia of the section.

Referring to article 5, Chapter V,

$$I_{zz} = I_{xx} + I_{yy}.$$

Since $I_{xx} = I_{yy}$ in a circular section,

$$I_{zz} = 2I_{xx}$$

$$\therefore \quad \boxed{J = 2I.}$$

For a solid circular section of diameter D,

$$\checkmark\ J = 2 \times \frac{\pi}{64} D^4 = \frac{\pi D^4}{32}$$

For a hollow circular section of external diameter D and internal diameter d,

$$\therefore J = 2 \times \frac{\pi}{64}\,(D^4 - d^4) = \frac{\pi}{32}\,(D^4 - d^4).$$

The relations (3) and (4) may be combined into one comprehensive statement as under:

$$\frac{q}{r} = \frac{f_s}{R} = \frac{T_r}{J} = \frac{N\theta}{l}. \quad \ldots\ldots\ldots\ldots (5)$$

The close analogy between this expression for torsional stresses in a shaft and that for bending stresses in a beam viz.,

$$\frac{p}{y} = \frac{f}{d/2} = \frac{M}{I}, \text{ will be noticed.}$$

For a shaft of solid circular section,

$$T_r = f_s \times \frac{J}{R} = f_s \times \frac{\pi D^4 \times 2}{32 \times D} = f_s \times \frac{\pi}{16} D^3.$$

For a shaft of hollow circular section,

$$T_r = f_s \times \frac{J}{R} = f_s \times \frac{\pi(D^4 - d^4)}{32} \div \frac{D}{2} = f_s \times \frac{\pi}{16} \cdot \frac{(D^4 - d^4)}{D}.$$

The expression $\frac{J}{R}$ corresponds to the expression $\frac{I}{d/2}$ or section-modulus Z for a beam and may be called the torsional section-modulus of the shaft section.

From (5) above, the angle of twist of the shaft section may be stated as,

$$\theta = \frac{T.l}{N.J} = \frac{f_s}{R} \cdot \frac{l}{N}, \text{ in radian measure.}$$

The angle of twist in a shaft under torsion corresponds to deflection in a beam under transverse loading and is an indication of the stiffness of the shaft. The expression $N.J$ corresponds to the expression $E.I$ in beam deflections and represents the torsional rigidity of the shaft. To ensure adequate stiffness of the shaft, the angle of twist is usually limited to about 1° in a length of 20 diameters.

3. Power transmitted: design of shafts: If a shaft rotating at N *r.p.m.* transmits H horse-power, the average torque T in lb-feet units acting on it will be given by,

$$H = \frac{\text{average couple} \times \text{angle turned through in one minute}}{33000}$$

$$= \frac{T \times 2\pi N}{33000}$$

$$\therefore \quad T = \frac{33000H}{2\pi N} \text{ lb-feet.}$$

For a shaft of a given section, transmitting a given horsepower at a given speed, the value of the torque transmitted can thus be worked out. The maximum shear stress f_s induced in the section will then be obtained by equating T to the moment of resistance, i.e., $f_s \times \frac{J}{R}$, taking care of the units employed.

If the diameter of the shaft has to be designed for transmitting a given H.P. at a given speed at a stress f_s not exceeding a given limit, we know the values of T and f_s in the expression $T = f_s \times \frac{J}{R}$ and we solve for the torsional section-modulus $\frac{J}{R}$ that is necessary.

It must be noted that T in the expression $\frac{33000H}{2\pi N}$ represents the average torque in lb-feet units acting on the shaft. In designing the section, we do not design it for the average torque but the maximum that is likely to come upon it, so that the permissible stress for the material is not exceeded. The value of the maximum torque is 30 to 40% higher than the average value.

Example 1.

A solid steel shaft, 3" diameter, transmits 120 H.P. at 180 R.P.M. Calculate the maximum intensity of shear stress induced and the angle of twist in degrees for a length of 25 feet. $N - 12 \times 10^6$ *lb/in²*.

$$T = \frac{33000 \times 120}{2\pi \times 180} = \frac{33000 \times 120 \times 7}{2 \times 22 \times 180} = 3500 \text{ lb-feet,}$$

or 42000 lb-inches.

Equating,

$$T = f_s \times \frac{\pi}{16} D^3$$

$$42000 = f_s \times \frac{\pi}{16} \times 27$$

$$\therefore \quad f_s = 7921 \text{ lb/in}^2.$$

$$\theta = \frac{f_s}{R} \times \frac{l}{N} = \frac{7921}{1{\cdot}5} \times \frac{300}{12 \times 10^6} = {\cdot}1320 \text{ radian} = 7°34'.$$

Example 2.

Find the H.P. transmitted by a shaft 2″ in diameter at 150 R.P.M. if the maximum permissible shear is 9000 lb/in².

Moment of resistance at 9000 lb/in².

$$= f_s \times \frac{\pi}{16} D^3 = 9000 \times \frac{\pi}{16} \times 8 = 4500\pi \text{ lb-inches}$$

$$= 375\pi \text{ lb-feet.}$$

$$\text{H.P. transmitted} = \frac{T \times 2\pi N}{33000} = \frac{375\pi \times 2\pi \times 150}{33000} = 33{\cdot}66.$$

Example 3.

A solid steel shaft has to transmit 80 H.P. at 200 R.P.M. The maximum torque transmitted in each revolution exceeds the mean by 30%. If the shearing stress is not to exceed 12000 lb/in²., find a suitable diameter for the shaft.

$$\text{Average torque } T = \frac{80 \times 33000}{2\pi \times 200} = 2100 \text{ lb-feet.}$$

$$T_{max} = 1{\cdot}3T = 1{\cdot}3 \times 2100 = 2730 \text{ lb-feet.}$$

$$= 32760 \text{ lb-inches.}$$

$$\therefore \quad 32760 = 12000 \times \frac{\pi}{16} \times D^3$$

$$\text{or} \quad D^3 = \frac{43{\cdot}68}{\pi} = 13{\cdot}9$$

$$\therefore \quad D = 2{\cdot}404'', \text{ say, } 2\tfrac{1}{2}''.$$

Example 4.

A hollow steel shaft has to transmit 750 H.P. at 90 R.P.M. The maximum twisting moment may exceed the mean by 40%. Design a suitable section if the permissible shear is 12000 lb/in². The thickness of metal is to be 1″.

Let D be the external diameter of the shaft in inches; the internal diameter $d = (D - 2)$ inches.

$$\text{Average torque } T = \frac{750 \times 33000}{2\pi \times 90} = 43750 \text{ lb-feet.}$$

$$= 525000 \text{ lb-inches.}$$

$$T_{max} = 1{\cdot}4 \;\; T = 1{\cdot}4 \times 525000$$

$$= 735000 \text{ lb-inches.}$$

$$\text{Moment of resistance} = f_s \times \frac{J}{R} = f_s \times \frac{\pi\,(D^4 - d^4)}{16D}$$

$$= \frac{12000\pi \;\; (2D^2 - 4D + 4) \;\; (4D \quad 4)}{16D}$$

$$= \frac{6000\pi \;\; (D \quad 1) \;\; (D^2 \quad 2D + 2)}{D}$$

Equating,

$$\frac{6000\pi \;\; (D \quad 1) \;\; (D^2 \quad 2D + 2)}{D} - 735000$$

$$\therefore \quad D^3 - 3D^2 + 4D - 2 = 39D$$

$$\therefore \quad D^3 - 3D^2 - 35D - 2 = 0$$

$D = 7{\cdot}6''$ satisfies the equation.

$$\therefore \quad d = 5{\cdot}6''.$$

A shaft 8″ external diameter and 6″ internal diameter will be satisfactory.

Example 5.

A shaft running at 180 R.P.M. has to transmit 130 H.P. The shaft must not be stressed beyond 9000 lb/in²., and must not twist more than 1° in a length of 10 feet. Select a suitable diameter, taking $N = 6000$ *tons/in².*

$$T = \frac{33000 \times 130}{2\pi \times 180} = 3792 \text{ lb-feet or } 45500 \text{ lb-inches.}$$

For strength,

$$T = f_s \times \frac{\pi}{16} D^3$$

$$\therefore \quad 45500 = \frac{\pi}{16} D^3 \times 9000$$

$\therefore \quad D^3 = 25{\cdot}74$

and $\quad D = 2{\cdot}952''$.

For stiffness,

$$\theta = \frac{T.l}{N.J}$$

$$\because \quad \frac{\pi}{180} = \frac{45500 \times 120 \times 32}{6000 \times 2240 \times \pi D^4}$$

$\therefore \quad D^4 = 236{\cdot}9$

and $\quad D = 3{\cdot}932''$.

4″ will be a suitable size for the shaft.

Example 6.

3000 H.P. has to be transmitted at 60 R.P.M. If the safe allowable stress is 12000 lb/in²., find the necessary diameter for a solid shaft of circular section. If a hollow circular section is used with its internal diameter $\frac{3}{4}$ the external diameter, calculate the saving in weight per foot length of the shaft. Density of the material = 480 lb. per cubic foot.

$$T = \frac{3000 \times 33000}{2\pi \times 60} \times 12 = 3150000 \text{ lb-inches.}$$

For a solid section,

$$3150000 = 12000 \times \frac{\pi}{16} D^3$$

$\therefore \quad D^3 = 1337$

$\therefore \quad D = 11{\cdot}02''$.

With a hollow circular section in which $d = \frac{3}{4} D$ we have,

$$\frac{\pi}{16} \frac{(D^4 - d^4)}{D} = \frac{\pi}{16} \left(1 + \frac{9}{16}\right) \left(1 - \frac{9}{16}\right) D^3$$

$$= \frac{175\pi}{16^3} . D^3.$$

$$\therefore \quad 3150000 = \frac{12000 \times 175\pi}{16 \times 16 \times 16} \times D^3$$

$\therefore \quad D^3 = 1955$

and $D = 12{\cdot}5''$

$\therefore \quad d = \frac{3}{4}D = 9{\cdot}375''$.

Weight per foot length of solid shaft is,

$$w_1 = \frac{\pi}{4} \times \frac{11{\cdot}02^2}{144} \times 1 \times 480 = 318 \text{ lb-wt.}$$

Weight per foot length of hollow shaft is,

$$w_2 = \frac{\pi}{4} \times \frac{21{\cdot}875 \times 3{\cdot}125}{144} \times 1 \times 480 = 179 \text{ lb-wt.}$$

Saving per foot length of shaft is.

$$w_1 - w_2 = 318 \quad 179 = 139 \text{ lb-wt.}$$

$$\text{Percentage saving} = \frac{139}{318} \times 100 = 43{\cdot}71\%.$$

As is to be expected, a hollow shaft is better than a solid one because the material near the neutral axis in the case of a solid shaft is wasted.

Example 7.

A steel shaft, 1" diameter and 3 feet long is rigidly fixed at the ends. A twisting moment of 200 lb-feet is applied to it at a distance of 9" from one end. Calculate the fixing couples at the ends, the maximum shear stress induced and the angle of twist of the section where the twisting moment is applied. $N = 12 \times 10^6$ *lb/in*^{2}.

Let l be the length of the shaft—in this case, 36"—and let the section, at which the couple T of 200 lb-feet is applied, divide it into segments l_1 and l_2. If T_1 and T_2 are the fixing couples at the supports,

$$T_1 + T_2 = T \quad \dots\dots\dots\dots\dots\dots (1)$$

The angle of twist θ at the section at which the couple is applied, is the same whether it is considered as part of segment l_1 or of segment l_2. We have,

$$\theta = \frac{T_1 l_1}{NJ} = \frac{T_2 l_2}{NJ}$$

i.e. $T_1 l_1 = T_2 l_2 \quad \dots\dots\dots\dots\dots\dots (2)$

Between equations (1) and (2), we solve for T_1 and T_2.

In this example, we have,

$$T_1 + T_2 = 200 \text{ lb-feet.}$$

$$27T_1 = 9T_2$$

i.e. $$3T_1 = T_2$$

$\therefore$ $$T_1 = 50 \text{ lb-feet.}$$

and $$T_2 = 150 \text{ lb-feet.}$$

For the maximum shear stress in the segment l_2,

$$150 \times 12 = f_s \times \frac{\pi}{16} \times 1^3$$

$$f_s = 9166 \text{ lb-/in}^2.$$

$$\text{The angle of twist } \theta = \frac{f_s}{R} \times \frac{l}{N}$$

$$= \frac{9166}{\frac{1}{2}} \times \frac{9}{12 \times 10^6} = \cdot 01375 \text{ radian}$$

$$= \cdot 7877^\circ.$$

4. **Keys and couplings:** The pulley transmitting power is keyed to the shaft and if the key is not able to take up the twisting moment transmitted across it, it will shear off. If there is a joint in the length of the shaft by means of a coupling, the coupling must be strong enough to transmit the twisting moment from one length of shaft to the other, coupled to it. Otherwise, the bolts of the coupling may shear off.

Let l and b be the length and width of the key; let the number of bolts of diameter d_b connecting the flanges of the coupling be n, arranged symmetrically at a radius R_b. If a twisting moment T is being transmitted by a shaft of diameter d (radius r), the shear stresses f_s, f_k and f_b set up in the shaft, key and bolt sections respectively can be easily calculated.

The resistance set up by the key section is $f_k \times l \times b$ at radius r of the shaft. The moment transmitted by the key is therefore, $T = f_k \times l \times b \times r$.

The shear force induced in a bolt section is $f_b \times \frac{\pi}{4} \times d^2_b$

acting at radius R_b. The moment transmitted by each bolt $= f_b \times \frac{\pi}{4} d^2{}_b \times R_b$.

The total moment transmitted by n bolts in the coupling is, therefore,

$$T = n \times f_b \times \frac{\pi}{4} d^2{}_b \times R_b.$$

We have, therefore,

$$T = f_s \times \frac{\pi}{16} d^3 = f_k \times l \times b \times r = f_b \times n \times \frac{\pi}{4} d^2{}_b \times R_b.$$

Example 8.

A 3″ diameter shaft transmits 120 H.P. at 150 R.P.M. A flanged coupling is keyed to the shaft by means of a key 5″ long and 1″ wide. The coupling has 6 bolts of $\frac{3}{4}$″ diameter symmetrically arranged along a bolt circle of 10″ diameter. Calculate the shear stresses in the shaft, the key and the bolts of the coupling.

$$T = \frac{120 \times 33000}{2\pi \times 150} \times 12 = 50400 \text{ lb-inches.}$$

$$50400 = f_s \times \frac{\pi}{16} \times 27 = f_k \times 5 \times 1 \times \tfrac{3}{2} = f_b \times 6 \times \cdot 44 \times 5.$$

Solving, we have,

$f_s = 9500$ lb/in^2.

$f_k = 6720$ lb/in^2.

$f_b = 3820$ lb/in^2.

5. Combined bending and torsion: In the previous articles, we have studied the stresses and strains set up by the cross-section of a circular shaft to resist the action of a twisting moment on it. The section has usually also to resist a bending moment due to the weight of the shaft and pulleys between its support-bearings and also due to the pulls on the belts driving the pulleys.

Let M and T respectively be the bending moment and twisting moment on the cross-section of a shaft of diameter d. Let us investigate the resulting stresses and strains.

Consider a point at radius r and at a distance y above or below the neutral axis XX. The effect of the bending moment M at the point is a direct stress p given by,

$$p = \frac{M.y}{I_{xx}}.$$

The effect of the twisting moment T at the point is a state of simple shear given by,

$$q = \frac{T \times r}{J}.$$

At the point under consideration there is, thus, a state of direct stress p accompanied by a state of simple shear q. This case has been studied in Chapter II. The position of the principal planes, through the point, is given by, $\tan 2\theta = \frac{2q}{p}$ and the values of the principal stresses p_1 and p_2 are obtained from,

$$p_1 = \frac{p}{2} + \sqrt{\frac{p^2}{4} + q^2}$$

and $p_2 = \frac{p}{2} - \sqrt{\frac{p^2}{4} + q^2}$.

Thus, we know the combined effect of M and T at any point in the cross-section. There are, however, only two points in the section viz., D and E [fig. 189(*a*)] on which the effect of each is a maximum. The bending stress f_b at these points is given by $M = f_b \times Z = f_b \times \frac{\pi}{32}d^3$, compressive at D and tensile at E; while the shear stress f_s is obtained from

$$T = f_s \times \frac{\pi}{16}d^3.$$

At D or E, then, the state of stress consists of a direct stress $f_b = \frac{32M}{\pi d^3}$, accompanied by a state of simple shear of intensity $f_s = \frac{16T}{\pi d^3}$.

The position of the principal planes through the point is obtained from,

$$\tan 2\theta = \frac{2f_s}{f_b} = \frac{T}{M}$$

The principal stresses f_1 and f_2 are given by,

$$f_1 = \frac{f_b}{2} + \sqrt{\frac{f_b^2}{4} + f_s^2} = \frac{16}{\pi d^3}\left\{M + \sqrt{M^2 + T^2}\right\}$$

$$\text{and } f_2 = \frac{f_b}{2} - \sqrt{\frac{f_b^2}{4} + f_s^2} = \frac{16}{\pi d^3}\left\{M \quad \sqrt{M^2 + T^2}\right\}$$

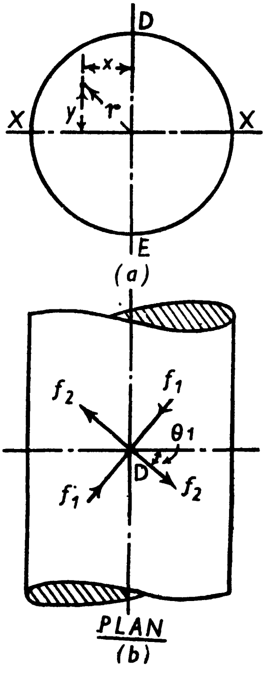


FIG. 189

The position of the principal planes and the principal stresses f_1 and f_2 acting at D have been marked in *plan* in fig. 189(*b*). It will be noted that f_1 and f_2 are unlike stresses.

The selection of a suitable diameter to resist given values of M and T may be made on one of the following hypotheses:

(i) *Principal stress:*

If the maximum direct stress is the criterion for design, f_1 must not exceed the working stress f for the material. We have,

$$f_1 = \frac{16}{\pi d^3}\left\{ M + \sqrt{M^2 + T^2}\right\} = f$$

$\therefore \quad M + \sqrt{M^2 + T^2} = f \times \frac{\pi d^3}{16}$, where f is the permissible direct stress for the material.

(ii) *Maximum principal strain:*

The major principal strain is,

$$e = \frac{f_1}{E} - \frac{f_2}{mE} = \frac{1}{E}\left\{\frac{f_b}{2}(1 - \frac{1}{m}) + (1 + \frac{1}{m})\sqrt{\frac{f_b^2}{4} + f_s^2}\right\}$$

$$\therefore \quad Ee = \frac{16}{\pi d^3}\left\{ M(1 \quad \frac{1}{m}) + (1 + \frac{1}{m})\sqrt{M^2 + T^2}\right\}.$$

If the stress, which, acting alone, produces the same maximum strain, is not to exceed the working stress f for the material, we have,

$$E.e = \frac{16}{\pi d^3}\left\{M (1 - \frac{1}{m}) + (1 + \frac{1}{m})\sqrt{M^2 + T^2}\right\} = f\cdot$$

or $\quad M(1 \quad \frac{1}{m}) + (1 + \frac{1}{m})\sqrt{M^2 + T^2} = f \times \frac{\pi}{16}d^3.$

(iii) *Maximum shear stress:*

The maximum shear intensity f_s' is given by,

$$f_s' = \frac{f_1 - f_2}{2} = \sqrt{\frac{f_b^2}{4} + f_s^2} = \frac{16}{\pi d^3}\sqrt{M^2 + T^2}.$$

If the criterion of design is the maximum permissible shear stress, we have,

$$\sqrt{M^2 + T^2} = f_s' \times \frac{\pi}{16}d^3.$$

(iv) *Maximum strain energy:*

The strain energy in terms of f_1 and f_2 is,

$\left\{\frac{f_1^2}{2E} + \frac{f_2^2}{2E} - \frac{2f_1 f_2}{2mE}\right\}$. If this is not to exceed $\frac{f^2}{2E}$, the strain energy at the permissible stress f, we have,

$$f^2 = f_1^2 + f_2^2 - \frac{2f_1f_2}{m} = f_b^2 + \frac{2\ (m+1)}{m}.\ f_s^2$$

$$\therefore\ f^2 = \left(\frac{32M}{\pi d^3}\right)^2 + \frac{2\ (m+1)}{m}\left(\frac{16T}{\pi d^3}\right)^2$$

$$= \left(\frac{16}{\pi d^3}\right)^2 \left\{4M^2 + \frac{2\ (m+1)}{m}.\ T^2\right\}$$

$$\therefore\ \sqrt{4M^2 + \frac{2\ (m+1)}{m}.\ T^2} = f \times \frac{\pi d^3}{16}.$$

Example 9.

At a certain cross-section of a shaft 4″ in diameter, there is a bending moment of 20 ton-inches and a twisting moment of 30 ton-inches. Calculate the maximum direct stress induced in the section and specify the position of the plane on which it acts. If Poisson's ratio is ·3, find the stress which, acting alone, will produce the same maximum (*i*) *strain* (*ii*) *strain energy.*

$$f_s = \frac{16T}{\pi d^3} = \frac{16 \times 30}{64\pi} = 2{\cdot}387 \text{ tons/in}^2.$$

$$f_b = \frac{32M}{\pi d^3} = \frac{32 \times 20}{64\pi} = 3{\cdot}183 \text{ tons/in}^2.$$

The principal planes will be given by,

$$\tan 2\theta = \frac{2f_s}{f_b} = \frac{T}{M} = \frac{30}{20} = 1{\cdot}5$$

$$\therefore\ 2\theta = 56°18'$$

$\therefore\ \theta_1 = 28°9'$ or $61°51'$ with the axis of the shaft

and $\theta_2 = 118°9'$ or $28°9'$ with the axis of the shaft.

$$f_1 = \frac{f_b}{2} + \sqrt{\frac{f_b^2}{4} + f_s^2} = 1{\cdot}592 + \sqrt{1{\cdot}592^2 + 2{\cdot}387^2}$$

$$= 1{\cdot}592 + 2{\cdot}869 = 4{\cdot}461 \text{ tons/in}^2.$$

$$f_2 = \frac{f_b}{2} - \sqrt{\frac{f_b^2}{4} + f_b^2} = 1{\cdot}592 - 2{\cdot}869 = -1{\cdot}277 \text{ tons/in}^2$$

(i) Maximum strain is,

$$e = \frac{f_1}{E} - \frac{f_2}{mE}$$

$$\therefore \quad Ee = f_1 - \frac{f_2}{m} = 4{\cdot}461 + {\cdot}3 \times 1{\cdot}277$$
$$= 4{\cdot}844 \text{ tons/in}^2.$$

(ii) $$\frac{f^2}{2E} = \frac{f_1^2}{2E} + \frac{f_2^2}{2E} - \frac{2f_1f_2}{2mE}$$

$$\therefore \quad f^2 = f_1^2 + f_2^2 - \frac{2f_1f_2}{m}$$
$$= (4{\cdot}461)^2 + (1{\cdot}277)^2 + 2 \times 4{\cdot}461 \times 1{\cdot}277 \times {\cdot}3$$
$$= 19{\cdot}90 + 1{\cdot}63 + 3{\cdot}42$$
$$= 24{\cdot}95$$
$$\therefore \quad f = 4{\cdot}995 \text{ tons/in}^2.$$

Example 10.

A circular shaft transmits 80 H.P. at 120 R.P.M. It is supported in bearings 10 feet apart and at 3 feet from one bearing, it carries a pulley exerting a transverse load of 8000 lb-wt. on the shaft. Determine a suitable diameter for the shaft if, (i) the maximum direct stress is not to exceed 6 tons/in^2., (ii) the maximum intensity of shear stress is not to exceed 3 tons/in^2., (iii) the stress which, acting alone, would produce the same maximum strain is not to exceed 6 tons/in^2., (iv) the direct stress which acting alone would produce the same maximum strain energy is not to exceed 6 tons/in^2. $\frac{1}{m} = {\cdot}25$.

$$T = \frac{80 \times 33000}{2\pi \times 120 \times 2240} \times 12 = 18{\cdot}75 \text{ ton-inches.}$$

$$M = \frac{8000 \times 3 \times 7}{10 \times 2240} \times 12 = 90 \text{ ton-inches.}$$

(i) The major principal stress f_1 is given by,

$$M + \sqrt{M^2 + T^2} = f_1 \times \frac{\pi}{16} d^3$$

$$\therefore \quad 90 + \sqrt{90^2 + 18{\cdot}75^2} = 90 + 91{\cdot}93 = 181{\cdot}93 = f_1 \times \frac{\pi}{16}d^3.$$

If f_1 is permitted to go upto 6 tons/in^2.,

$$\frac{\pi}{16}d^3 \times 6 = 181{\cdot}93.$$

$\therefore \quad d^3 = 154{\cdot}4$

and $\quad d = 5{\cdot}364''$.

(ii) If f_s' is the maximum intensity of shear stress,

$$\sqrt{M^2 + T^2} = f_s' \times \frac{\pi}{16} d^3$$

$$\therefore \qquad 91{\cdot}93 = \frac{\pi}{16} d^3 . f_s' .$$

If f_s' is permitted to reach 3 tons/in^2., we have,

$$\frac{\pi}{16} d^3 \times 3 = 91{\cdot}93$$

$\therefore \qquad d^3 = 156{\cdot}1$

and $\qquad d = 5{\cdot}384''$.

(iii) If e is the major principal strain,

$$E.e \times \frac{\pi}{16} d^3 = \left(1 - \frac{1}{m}\right) M + \left(1 + \frac{1}{m}\right) \sqrt{M^2 + T^2}$$

If Ee is allowed to reach 6 tons/in^2., we have,

$$\frac{\pi}{16} d^3 \times 6 = \tfrac{3}{4} \times 90 + \tfrac{5}{4} \times 91{\cdot}93$$

$$= 67{\cdot}5 + 114{\cdot}91 = 182{\cdot}41$$

$\therefore \qquad d^3 = 154{\cdot}7$

and $\qquad d = 5{\cdot}369''$.

(iv) If f is the direct stress which, acting alone, will produce the same maximum strain energy,

$$\frac{\pi}{16} d^3 . f = \sqrt{4M^2 + \frac{2\ (m+1)}{m} . \ T^2}$$

$$= \sqrt{4 \times 90^2 + \tfrac{5}{2}\ (18{\cdot}75)^2}$$

$$= 182{\cdot}4.$$

If f reaches 6 tons/in^2.,

$$d^3 = \frac{182{\cdot}4 \times 16}{6\pi} = 154{\cdot}7$$

$\therefore \qquad d = 5{\cdot}369''$.

The diameter of the shaft will be $5\frac{1}{2}''$.

6. Combined bending and torsion and axial thrust: Cases arise — such as the propeller shafts of ships, in which, besides resisting a bending moment and a torque, a shaft is subjected to direct thrust. When this is so, the direct stresses due to the bending moment and the axial thrust will be combined into a single resultant. At a point in the section of the shaft, there will thus be a direct stress equal to this resultant accompanied by a state of simple shear due to the twisting moment. Of the two points D and E in fig. 189, D is now the only critical point, since, the direct stresses due to the bending moment and the axial thrust are both compressive at D. The principal planes and principal stresses may be determined, as usual.

Example 11.

A propeller shaft, 12″ external diameter, 6″ internal diameter, transmits 2400 H.P. at 80 R.P.M. There is, at the same time, a bending moment of 40 ton-feet and an end thrust of 25 tons. Find (i) the maximum principal stresses and their planes (ii) the maximum shear stress (iii) the stress, which acting alone, will produce the same maximum strain. Take $\frac{1}{m} = \cdot 3$.

$$\text{Area of section} = \frac{\pi}{4}(12^2 - 6^2) = 84{\cdot}9 \text{ in}^2.$$

$$Z \text{ of the section} = \frac{\pi}{32} \cdot \frac{D^4 - d^4}{D} = \frac{\pi}{32} \times \frac{180 \times 108}{12}$$

$$= 159{\cdot}1 \text{ in}^3.$$

$$\text{Torsional modulus of section} = \frac{\pi}{16} \times \frac{D^4 - d^4}{D} = 318{\cdot}2 \text{ in}^3.$$

$$\text{Now, } p_o = \frac{P}{A} = \frac{25}{84{\cdot}9} = {\cdot}2944 \text{ ton/in}^2.$$

$$p_b = \frac{M}{Z} = \frac{480}{159{\cdot}1} = 3{\cdot}016 \text{ tons/in}^2.$$

$$\text{Maximum direct stress} = {\cdot}2944 + 3{\cdot}016$$

$$= 3{\cdot}31 \text{ tons/in}^2.$$

The twisting moment transmitted

$$= \frac{2400 \times 33000}{2\pi \times 80 \times 2240} \times 12 = 843{\cdot}75 \text{ ton-inches.}$$

The maximum shear stress will be given by,

$$843{\cdot}75 = 318{\cdot}2 \times f_s$$

$$\therefore \quad f_s = 2{\cdot}653 \text{ tons/in}^2.$$

(i) The position of the principal planes is obtained from,

$$\tan 2\theta = \frac{2 \times 2{\cdot}653}{3{\cdot}31} = 1{\cdot}603$$

$$\therefore \quad 2\theta = 58°2'$$

$$\therefore \quad \theta_1 = 29°1'$$

$$\text{and} \quad \theta_2 = 119°1'.$$

$$f_1 = 1{\cdot}655 + \sqrt{1{\cdot}655^2 + 2{\cdot}653^2}$$

$$= 1{\cdot}655 + 3{\cdot}127 = 4{\cdot}782 \text{ tons/in}^2\text{., compressive.}$$

$$f_2 = 1{\cdot}655 - 3{\cdot}127 = \quad 1{\cdot}472 \text{ tons/in}^2.$$

i.e. $1{\cdot}472$ tons/in^2., tensile.

(ii) The maximum shear intensity

$$= \sqrt{1{\cdot}655^2 + 2{\cdot}653^2} = 3{\cdot}127 \text{ tons/in}^2.$$

(iii) Maximum strain is,

$$e = \frac{f_1}{E} - \frac{f_2}{mE}$$

$$\therefore \quad Ee = f_1 - \frac{f_2}{m} = 4{\cdot}782 + {\cdot}3 \times 1{\cdot}472$$

$$= 5{\cdot}224 \text{ tons/in}^2.$$

7. Torsional resilience of shafts: When the cross-section of a member is subjected to a shear stress of uniform intensity q, we have seen that the strain energy stored in the member is $\frac{q^2}{2N} \times$ volume. In the case of a cylindrical shaft, the torsional stress varies from zero at the central axis to a maximum f_s at the surface. At radius r, the shear intensity is given by $q = \frac{f_s}{R} \times r$. If we consider an elementary ring of thickness δr over which the shear intensity is sensibly uniform, the energy stored by the elementary shell will be $\frac{q^2}{2N} \cdot \delta a . l$, where $\delta a = 2\pi r \delta r$ and l is the length of the shaft.

Substituting $f_s \frac{r}{R}$ for q, the expression becomes,

$$\frac{f_s^2}{2N.R^2} \times 2\pi l \times r^3 dr.$$

Integrating over a range from $r = 0$ to $r = \frac{D}{2}$ for a solid shaft of diameter D, the shear energy stored in the shaft is given by,

$$\frac{\pi f_s^2}{NR^2} . l \int_0^{\frac{D}{2}} r^3 dr = \frac{f_s^2}{N} \times \frac{\pi D^4}{64 R^2} \times l = \frac{f_s^2}{4N} \times \frac{\pi}{4} D^2 \times l$$

$$= \frac{f_s^2}{4N} \times \text{volume of the shaft.}$$

For a hollow shaft of external diameter D and internal diameter d, the range of integration will be from $\frac{d}{2}$ to $\frac{D}{2}$.

$$\therefore \quad \text{Shear energy} = \frac{\pi f_s^2}{NR^2} . l \int_{\frac{d}{2}}^{\frac{D}{2}} r^3 dr = \frac{\pi f_s^2 . l \ (D^4 - d^4)}{64 N R^2}$$

$$= \frac{f_s^2}{4N} \times \frac{\pi \ (D^4 \quad d^4)}{4D^2} \times l = \frac{f_s^2}{4N} \times \frac{D^2 + d^2}{D^2} \times \frac{\pi}{4} (D^2 - d^2) \ l$$

$$= \frac{f_s^2}{4N} \times \frac{(D^2 + d^2)}{D^2} \times \text{volume of the shaft.}$$

8. Shafts of non-circular sections: The theory of torsion given in the preceding articles is applicable only to shafts of circular section. For non-circular sections, an exact study of the stresses and strains produced by a twisting moment is rather complicated. The problem was studied by St. Venant who obtained values of $\underline{N.J}$ for a number of non-circular sections. The results may be summarised as under:

Section	Moment of resistance	Angle of twist in radians
Square: length of side, s	$\cdot 208 s^3 . f_s$	$\dfrac{7\cdot 11\ T.l}{N.s^4}$
Elliptic section: Major axis, d_1 Minor axis, d_2	$\dfrac{\pi}{16}.d_1.d_2^2 f_s$	$\dfrac{16}{\pi}.\dfrac{(d_1^2+d_2^2)}{d_1^3.d_2^3}.\dfrac{T.l}{N}$
Rectangular section: long side, a short side, b	$\dfrac{a^2 b^2}{3a+1\cdot 8b}.f_s$	$k.\dfrac{a^2+b^2}{a^3 b^3}.\dfrac{T.l}{N}$,

where $k = (3\cdot 645 - \cdot 06\frac{a}{b})$, approximately.

For any solid symmetrical sections, including rectangles, the angle of twist θ is, approximately,

$\theta = \dfrac{40 J}{A^4}.\dfrac{Tl}{N}$, where A is the area of section and J is the polar moment of Inertia.

The maximum intensity of stress f_s occurs at the point in the perimeter of the section at the shortest radius vector.

Example 12.

A shaft of square section running at 180 R.P.M., transmits power to a crane lifting 10 tons at a speed of 25 feet per minute. If the efficiency of the crane gearing is 64%, calculate the size of the shaft if the maximum permissible shear stress is 5500 lb/in^2. Calculate also the angle of twist in the shaft for a length of 10 feet if $N = 12 \times 10^6$ lb/in^2.

Work to be supplied by the shaft per minute

$$= \frac{10 \times 2240 \times 25}{\cdot 64} = 875000 \text{ ft-lbs.}$$

$$\therefore \quad T \times 2\pi \times 180 = 875000$$

$$\therefore \quad T = \frac{875000}{2\pi \times 180} \times 12 = 9283 \text{ lb-inches.}$$

$$
\begin{aligned}
\text{Now,}\qquad & {\cdot}208\, f_s . s^3 = T\\
\therefore\quad {\cdot}208 \times 5500 \times & s^3 = 9283\\
\therefore\qquad & s^3 = 8{\cdot}114\\
\text{or}\qquad & s = 2{\cdot}009'',\ \text{say, } 2'',\\
& \theta = \frac{7{\cdot}11 T.l}{N.s^4} = \frac{7{\cdot}11 \times 9283 \times 120}{12 \times 10^6 \times 16} \times \frac{180}{\pi}\\
& = 2{\cdot}363^\circ.
\end{aligned}
$$

9. Closely coiled helical springs: axial loading: Consider a shaft of diameter *d* and length *l*, fixed at one end and subjected at the other end to a tangential effort *W* acting at a radius *R*.

To find the stresses set up in the section of the shaft and the linear displacement of the point of application *A* at which the effort *W* is applied. See fig. 190.

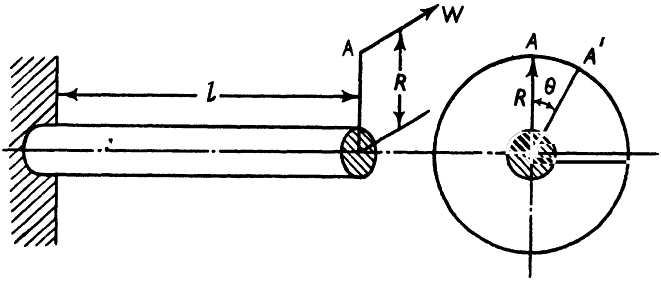


FIG. 190

Twisting moment on all sections of the shaft = *WR*.

$T = WR = f_s . \frac{\pi}{16} d^3$. This gives the maximum intensity of shear stress f_s induced at the surface at any section of the shaft.

The angle of twist of the section at the free end is,

$$\theta = \frac{T.l}{NJ} = \frac{WRl}{N.J}$$

The displacement *AA'* of the point of application of the effort will be,

$$\delta = R.\theta = \frac{WR^2.l}{N.J} = \frac{32W.R^2.l}{N.\pi d^4}$$

The amount of energy stored in the shaft will be given by,

$$\text{Resilience} = \tfrac{1}{2}\,W.\delta = \frac{16W^2R^2.l}{N.\pi d^4} = \frac{f_s^2}{4N} \times \frac{\pi}{4}\,d^2 \times l,$$

substituting $WR = f_s.\frac{\pi}{16}d^3$.

If a straight shaft of length l and diameter d is coiled to a mean radius of R with n turns, it becomes a closely coiled spring, for which,

$l = 2\pi R.n$, $2\pi R$ being the length of one coil.

If W is the axial loading on the spring, the twisting moment on any section of the spring is $T = WR$.

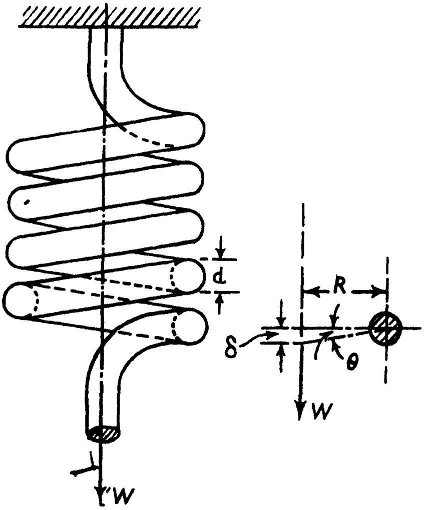


FIG. 191

The shear stress f_s induced in any section will be obtained from,

$$WR = f_s.\frac{\pi d^3}{16}$$

The angle of twist of the last section is,

$$\theta = \frac{T.l}{N.J} = \frac{WR.l.}{N.J}.$$

The deflection δ of the spring is, then,

$$\delta = R.\theta = \frac{WR^2.l}{N.J} = \frac{32W.R^2.l}{N.\pi d^4}$$

$$= \frac{64W.R^3.n}{N.d^4}, \text{ substituting } 2\pi R.n \text{ for } l.$$

The stiffness s of the spring will be the load per unit deflection.

$$\therefore \quad s = \frac{W}{\delta} = \frac{N.d^4}{n.64R^3}$$

Example 13.

A closely coiled helical spring is made out of a $\frac{3}{8}''$ diameter steel rod, the coils having 12 complete turns and a mean diameter of 4". It carries an axial pull of 50 lb-wt. Calculate the shear stress induced in the section of the rod. If $N = 12 \times 10^6$ lb/in²., calculate the deflection under the pull and the amount of energy stored in the spring during the extension.

$$T = W.R = 50 \times 2 = 100 \text{ lb-inches.}$$

$$\frac{\pi}{16}d^3 = \frac{\pi}{16}\left(\tfrac{3}{8}\right)^3 = \cdot 01035 \text{ in}^3.$$

Since $T = f_s.\frac{\pi}{16}d^3.$

$$100 = \cdot 01035 \times f_s$$

$$\therefore \quad f_s = 9658 \text{ lb/in}^2.$$

$$\delta = R.\theta = \frac{64W.R^3.n}{N.d^4} = \frac{64 \times 50 \times 8 \times 12}{12 \times 10^6 \times \left(\frac{3}{8}\right)^4} = 1\cdot 295''.$$

$$\text{Energy stored} = \tfrac{1}{2}\,W\delta = \tfrac{1}{2} \times 50 \times 1\cdot 295$$

$$= 32\cdot 375 \text{ inch-lbs.}$$

Example 14.

A truck weighing 2 tons and moving at 4 miles per hour has to be brought to rest by a buffer. Find how many springs, each of 18 coils, will be required to store the energy of motion during a compression of

9"? The spring is made out of 1" diameter steel rod coiled to a mean diameter of 8 inches. $N = 6000$ *tons/in*^{2}.

The K.E. of the truck to be absorbed by the buffer-springs

$$= \frac{W.v^2}{2g} = 2 \times \left(\frac{176}{30}\right)^2 \times \frac{1}{64} = \frac{968}{900} \text{ ft-tons.}$$

$$= 12{\cdot}91 \text{ inch-tons.}$$

The axial load P, for each spring, for a compression of 9" is given by,

$$9 = \frac{64P.R^3.n}{N.d^4} = \frac{64P \times 64 \times 18}{6000 \times 1}$$

$$\therefore \quad P = \frac{3000}{64 \times 64} \text{ ton.}$$

Energy stored by one spring

$$= \tfrac{1}{2}P.\delta = \tfrac{1}{2} \times \frac{3000}{64 \times 64} \times 9 = 3{\cdot}296 \text{ inch-tons.}$$

The number of springs required

$$= \frac{12{\cdot}91}{3{\cdot}296} = 3{\cdot}916, \text{ say, } 4.$$

Example 15.

A weight W of 5 cwt. is dropped on to a closely coiled compression spring with 18 coils. Calculate the height of drop before striking so that the spring is compressed by 9". Diameter of rod 1"; mean diameter of coils 8"; $N = 6000$ *tons/in*^{2}.

Let h inches be the height of drop and let P tons be the equivalent gradually applied load to produce the same compression.

$$\text{Then } 9 = \frac{64.P.R^3.n}{N.d^4} = \frac{64.P \times 64 \times 18}{6000 \times 1}$$

$$\therefore \quad P = \frac{3000}{64 \times 64} \text{ tons.}$$

Equating the energy supplied by the impact load to the energy stored,

$$W\ (h + \delta) = \tfrac{1}{2}P.\delta.$$

$$\therefore \qquad \tfrac{1}{4}(h+9) = \tfrac{1}{2} \times \frac{3000}{64 \times 64} \times 9$$

$$= 3{\cdot}296$$

$$\therefore \qquad h + 9 = 13{\cdot}184$$

or $$h = 4{\cdot}184''.$$

Example 16.

Find the maximum permissible axial load for a closely coiled spring made out of $\frac{1}{4}''$ square rod with 12 coils of 3″ mean diameter if the maximum shear stress is limited to 45000 lb/in^2. Calculate also the deflection under the load if $N = 12 \times 10^6$ lb/in^2.

$$T = {\cdot}208.s^3.f_s = {\cdot}208(\tfrac{1}{4})^3 \times 45000$$

$$= 146{\cdot}25 \text{ lb-inches.}$$

$$T = W.R.$$

$$\therefore \quad 1{\cdot}5W = 146{\cdot}25$$

or $$W = 97{\cdot}5 \text{ lb-wt.}$$

$$\theta = \frac{7{\cdot}11\, T.l}{N.s^4} = \frac{7{\cdot}11 \times 146{\cdot}25 \times \pi \times 3 \times 12}{12 \times 10^6 \times (\frac{1}{4})^4}$$

$$= 2{\cdot}51 \text{ radians.}$$

$$\delta = R.\theta = \tfrac{3}{2} \times 2{\cdot}51 = 3{\cdot}765''.$$

10. Closely coiled helical springs: axial twist: Let a closely coiled helical spring be fixed at one end and subjected at the other end to a twisting couple M about the central axis of the spring. Depending upon the *sense* of the couple, the coils will tend to bend or unbend, each cross-section resisting a *bending moment* equal to the couple M.

Let n be the number of coils of mean radius R. If the bending moment M increases the curvature, let the mean radius of the coil decrease from R to R' and let n' be the resulting number of coils. If l is the length of the wire forming the coiled spring, we have,

$$l = 2\pi R.n = 2\pi R'.n'.$$

Also $$\frac{M}{I} = E \times \text{change of curvature}$$

$$= E\left(\frac{1}{R'} - \frac{1}{R}\right) = \frac{2\pi E}{l}\ (n' - n)$$

$$\therefore \quad M = \frac{2\pi EI\ (n' - n)}{l},$$

where I is the usual moment of Inertia of the section about its neutral axis for bending.

If φ is the total angle of bend, in radians,

$$\varphi = 2\pi\ (n' \quad n) = \frac{M.l}{EI}$$

Differentiating,

$$\frac{d\varphi}{dl} = \frac{M}{EI}.$$

Therefore, the change of curvature or angle of bend per unit length is constant throughout and equal to $\frac{M}{EI}$.

For a wire of circular section of diameter d, the maximum bending stress f is given by,

$f = \frac{32M}{\pi d^3}$ and the angle φ is given by,

$\varphi = \frac{M.l}{EI} = \frac{64Ml}{E.\pi d^4} = \frac{128M.R.n}{Ed^4}$, in radians.

For a wire of square section of side s,

$f = \frac{6M}{s^3}$ and

$$\varphi = \frac{12M.l}{E.s^4} = \frac{24\pi.M.R.n}{E.s^4}.$$

The amount of strain energy stored in the spring will be $\frac{1}{2}M.\varphi$. In terms of the stress induced, it is $\frac{f^2}{8E} \times$ volume of the spring wire, for circular section and $\frac{f^2}{6E} \times$ volume, for rectangular sections, as has been proved before.

Example 17.

The spring in example 13, is subjected to an axial twist of 75 lb-inches. Calculate the increase in the number of turns and the bending stress induced in the section. $E = 30 \times 10^6$ *lb/in*^{2}.

$$I = \frac{\pi}{64}\,d^4 = \frac{\pi}{64}\,(\tfrac{3}{8})^4 = \cdot 0009705 \text{ in}^4.$$

$$Z = \frac{I}{d/2} = \cdot 005177 \text{ in}^3.$$

$$f = \frac{M}{Z} = \frac{75}{\cdot 005177} = 14490 \text{ lb/in}^2.$$

$$\varphi = \frac{M.l}{EI} = \frac{75 \times 2\pi \times 2 \times 12}{30 \times 10^6 \times \cdot 0009705} = 2\pi \times \cdot 06183 \text{ radians}$$

$$= 22{\cdot}26^\circ.$$

Since, $\varphi = 2\pi\,(n' \quad n)$,

$(n' - n) = \cdot 06183$

and $n' = 12{\cdot}06183$ turns.

11. Open coiled helical springs: (i) Axial loading:

Let α be the angle of the helix of an open coil helical spring of mean radius R, diameter of section d and number of turns n. If W is the axial load, its moment WR may be resolved into two components:

$T = WR \cos\alpha$, and

$M = WR \sin\alpha$ about the axes $X'X'$ and $Y'Y'$ at the centre O in the section of the wire (fig. 192). T will act as a twisting moment on the normal cross-section about the axis $X'X'$ and M will act as a bending moment about the axis $Y'Y'$, tending to unwind the coil.

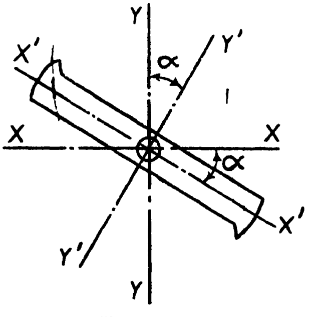


Fig. 192

If θ is the angle of twist due to T, $\theta = \dfrac{T.l}{NJ}$.

If φ is the angular rotation due to M, $\varphi = \dfrac{M.l}{EI}$.

The deflection δ of the spring under the axial load W can be obtained from,

$$\tfrac{1}{2} W.\delta = \tfrac{1}{2} T.\theta + \tfrac{1}{2} M.\varphi$$

or

$$W\delta = T\theta + M\varphi$$

Substituting for T and M, we get,

$$\delta = WR^2 l \left\{ \frac{\cos^2\alpha}{NJ} + \frac{\sin^2\alpha}{EI} \right\}$$

Since the length of the wire comprising the spring is,

$$l = 2\pi R.n.\ \sec\alpha,$$

$$\delta = 2\pi WR^3.n.\ \sec\alpha \left\{ \frac{\cos^2\alpha}{NJ} + \frac{\sin^2\alpha}{EI} \right\}$$

For a wire of circular section of diameter d,

$$I = \frac{\pi}{64} d^4 \text{ and } J = \frac{\pi d^4}{32}.$$

Substituting,

$$\delta = \frac{64 WR^3.n.\sec\alpha}{d^4} \left\{ \frac{\cos^2\alpha}{N} + \frac{2\sin^2\alpha}{E} \right\}$$

The angular twist θ' at the free end, about the axis OX will be given by,

$$\theta' = \frac{\delta}{R} = W.Rl. \left\{ \frac{\cos^2\alpha}{NJ} + \frac{\sin^2\alpha}{EI} \right\}$$

$$= \frac{64 W.R^2.n.\sec\alpha}{d^4} \left\{ \frac{\cos^2\alpha}{N} + \frac{2\sin^2\alpha}{E} \right\}$$

Similarly, the angular rotation φ' of the free end about the axis OY can be obtained by considering an element δl of the wire at O.

Due to $T = WR\cos\alpha$, $\delta\theta$ about the axis OX'

$$= \frac{T.\delta l}{NJ} = \frac{WR\cos\alpha.\delta l}{NJ}.$$

The component of this along the axis OY is,

$$\delta\theta \sin\alpha = \frac{WR \sin\alpha \cos\alpha . \delta l}{NJ}.$$

Due to $M = WR \sin\alpha$, $\delta\varphi$ about the axis OY'.

$$= \frac{M.\delta l}{EI} = \frac{WR \sin\alpha . \delta l}{EI}.$$

The effect of this is to unwind and therefore its component along OY may be stated as,

$$-\delta\varphi\cos\alpha = \frac{WR\sin\alpha \cos\alpha . \delta l}{EI}$$

Hence the resultant rotation $\delta\varphi'$ about OY due to both causes will be,

$$\delta\varphi' = \delta\theta \sin\alpha - \delta\varphi\cos\alpha = WR\sin\alpha \cos\alpha \ \delta l\left\{\frac{1}{NJ} - \frac{1}{EI}\right\}$$

i.e. $$\frac{\delta\varphi'}{\delta l} = WR \sin\alpha \cos\alpha \left\{\frac{1}{NJ} - \frac{1}{EI}\right\}$$

$$\therefore \quad \varphi' = WRl.\sin\alpha\cos\alpha\left\{\frac{1}{NJ} - \frac{1}{EI}\right\}.$$

Stresses in wire of circular section diameter d:

Due to $M = WR \sin\alpha$, the maximum bending stress f in the normal section of the wire is given by,

$$M = f.Z$$

or $$f = \frac{32WR\sin\alpha}{\pi d^3}$$

Due to $T = WR \cos\alpha$. the maximum shear stress f_s is given by,

$$T = f_s\frac{\pi}{16}d^3$$

or $$f_s = \frac{16WR \cos\alpha}{\pi d^3}.$$

The principal stresses f_1 and f_2 will be given by,

$$f_1 = \frac{f}{2} + \sqrt{\frac{f^2}{4} + f_s^2} = \frac{16WR}{\pi d^3}\left\{\sin\alpha + \sqrt{\sin^2\alpha + \cos^2\alpha}\right\}$$

$$= \frac{16WR}{\pi d^3}(1+\sin\alpha)$$

$$f_2 = \frac{f}{2} \quad \sqrt{\frac{f^2}{4}+f_s} = -\frac{16WR}{\pi d^3}(1 \quad \sin\alpha).$$

The maximum shear stress f_s' is given by,

$$f_s' = \frac{f_1-f_2}{2} = \frac{16WR}{\pi d^3}$$

(ii) Axial torque:

If an axial torque M about the axis of the coil is applied, its components will be $M\cos\alpha$ about the axis OY' and $M\sin\alpha$ about OX'. If φ' is the total rotation at the free end about the axis OY, we have,

$$\tfrac{1}{2}M.\varphi' = \tfrac{1}{2}\frac{(M\cos\alpha)^2 l}{EI} + \tfrac{1}{2}\frac{(M\sin\alpha)^2 l}{NJ}$$

$$\therefore \quad \varphi' = M.l\left\{\frac{\cos^2\alpha}{EI}+\frac{\sin^2\alpha}{NJ}\right\} = M \times 2\pi R.n\sec\alpha\left\{\frac{\cos^2\alpha}{EI}+\frac{\sin^2\alpha}{NJ}\right\}$$

The axial extension $\triangle$ caused by the couple may be deduced by resolving the rotations, as before.

$$\triangle = M.R\sin\alpha\cos\alpha.l\left(\frac{1}{NJ} \cdot \frac{1}{EI}\right)$$

$$= M.2\pi R^2.n.\sin\alpha\left(\frac{1}{NJ}-\frac{1}{EI}\right)$$

For a circular section of diameter d,

$$\varphi' = \frac{64MR.n\sec\alpha}{d^4}\left\{\frac{2\cos^2\alpha}{E}+\frac{\sin^2\alpha}{N}\right\} \text{ and}$$

$$\triangle = \frac{64MR^2.n\sin\alpha}{d^4}\left\{\frac{1}{N}-\frac{2}{E}\right\}$$

Example 18.

If the spring in example 13, is open coiled, the angle of the helix being 15°, calculate the deflection under the axial load of 50 lb., and the maximum intensities of direct and shear stresses induced in the section of the wire.

$T = W.R.\cos\alpha = 50 \times 2 \times \cdot 9659 = 96\cdot 59$ lb-inches.

$M = W.R.\sin\alpha = 50 \times 2 \times \cdot 2588 = 25\cdot 88$ lb-inches.

$$\therefore \quad \delta = \frac{64 \times 50 \times 2^3 \times 12 \times 1{\cdot}035}{(\frac{3}{8})^4}\left(\frac{{\cdot}9328}{12 \times 10^6} + \frac{2 \times {\cdot}067}{30 \times 10^6}\right)$$

$$= \frac{64 \times 400 \times 1{\cdot}035}{(\frac{3}{8})^4 \times 10^6} \times {\cdot}9864$$

$$= 1{\cdot}322''.$$

For the stresses,

$$f = \frac{M}{Z} = \left(\frac{25{\cdot}88}{{\cdot}005177}\right) = 5000 \text{ lb/in}^2.$$

$$f_s = \frac{96{\cdot}59}{{\cdot}01035} = 9333 \text{ lb/in}^2.$$

The principal stresses are given by,

$$f_1 = \frac{f}{2} + \sqrt{\frac{f^2}{4} + f_s^2} = 2500 + \sqrt{2500^2 + 9333^2}$$

$$= 2500 + 9662 = 12162 \text{ lb/in}^2.$$

$$f_2 = 2500 - 9662 = \quad 7162 \text{ lb/in}^2.$$

The maximum intensity of shear stress f_s' is given by

$$f_s' = \sqrt{\frac{f^2}{4} + f_s^2} = 9662 \text{ lb/in}^2.$$

Example 19.

In the above example, if the axial load of 50 lb. is replaced by an axial torque of 75 lb-inches, calculate the angle of rotation about the axis of the coil and the axial deflection.

$$\varphi' = \frac{64MR.n\sec\alpha}{d^4}\left\{\frac{2\cos^2\alpha}{E} + \frac{\sin^2\alpha}{N}\right\}$$

$$= \frac{64 \times 75 \times 2 \times 12 \times 1{\cdot}035}{(\frac{3}{8})^4}\left\{\frac{2 \times {\cdot}9328}{30 \times 10^6} + \frac{{\cdot}067}{12 \times 10^6}\right\}$$

$$= {\cdot}4087 \text{ radian or } 23{\cdot}42^\circ.$$

$$\triangle = \frac{64MR^2.n\sin\alpha}{d^4}\left\{\frac{1}{N} - \frac{2}{E}\right\}$$

$$= \frac{64 \times 75 \times 2^2 \times 12 \times {\cdot}2588}{(\frac{3}{8})^4}\left\{\frac{1}{12 \times 10^6} - \frac{2}{30 \times 10^6}\right\}$$

$$= {\cdot}0503''.$$

EXAMPLES XII

(1) Calculate the diameter of shaft required to transmit 60 H.P. at 120 r.p.m. if the maximum torque is likely to exceed the mean by 30%, for a maximum permissible shear stress of 8200 lb/in². Calculate also the angle of twist for a length of 6 feet. $\mathcal{N} = 12 \times 10^6$ lb/in². (3"; 1·772°.)

(2) A hollow circular shaft, 10" external diameter, thickness of metal 1", is transmitting power at 180 r.p.m. The angle of twist on a length of 10 feet was found to be ·72°. Calculate the H.P. transmitted and the maximum shear stress induced in the section. $\mathcal{N} = 6000$ tons/in². (2333; 3·142 tons/in².)

(3) A flanged coupling has six $1\frac{1}{8}''$ diameter bolts arranged symmetrically along a bolt circle of diameter 10". If the shaft is 4" diameter and stressed to 12000 lb/in²., calculate the shear stress in the bolts. (5026 lb/in².)

(4) A steel shaft of solid circular section has to transmit 500 H.P. at 210 r.p.m. The maximum shear stress is not to exceed 8000 lb/in². and the angle of twist must not be more than 1° in a length of 8 feet. Select a suitable diameter. $\mathcal{N} = 12 \times 10^6$ lb/in². (5·144".)

(5) A square shaft has to transmit 80 H.P. at 150 r.p.m. Calculate the section required if the permissible shear stress is 9000 lb/in². Calculate also the angle of twist on a length of 10 feet. $\mathcal{N} = 12 \times 10^6$ lb/in². (2·62" side; 2·313°.)

(6) A 1" diameter circular steel shaft is provided with enlarged portions *A* and *B* as shown in fig. 193. On to this enlarged portion, a steel tube $\frac{1''}{20}$ thick is shrunk. While the shrinking process is going on, the 1" shaft is held twisted by a couple of magnitude 50 lb-feet. When the tube is firmly set on the shaft, this twisting couple is removed. Calculate what twisting couple is left on the shaft, the shaft and the tube being made of the same material. (C. U.)
(465·1 lb-in.)

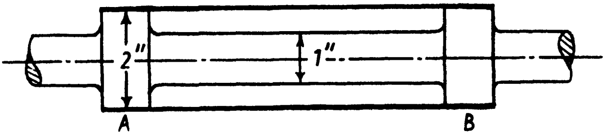


FIG. 193

(7) Compare the weight of a solid shaft with that of a hollow one to transmit a given horse power at a given speed with a given maximum shearing stress, the inside diameter of the hollow shaft being two-thirds of the outside diameter. (C. U.)

(Solid: Hollow:: 1·554: 1.)

(8) A 4″ diameter shaft is subjected to a bending moment M and a twisting moment T. The magnitude of the maximum principal stress due to these moments is 7 tons/in^2. If the maximum bending stress due to M is equal to the maximum shear stress due to T, evaluate M and T. ($M=27\cdot16$; $T=54\cdot32$ ton-inches.)

(9) An overhung engine crank measures 15″ from the centre line of the crank shaft to the centre line of the crank pin. The distance from the centre of the crank pin to the centre of the bearing is 9″, measured horizontally and the maximum thrust transmitted by the connecting rod at the crank pin is 8000 lb-wt. Calculate the diameter of the crank shaft if the maximum shear stress in it is 12000 lb/in^2.

(3·902″. say, 4″.)

(10) A shaft, 4″ diameter, transmits 120 H.P. at 150 r.p.m. It is supported in bearings 10 feet apart, and carries a pulley exerting a transverse pull of $1\frac{1}{2}$ tons at 4 feet from one support. Calculate the maximum direct stress induced in the cross-section of the shaft.

(7·313 tons/in^2.)

(11) A propeller shaft 8″ in diameter transmits 3000 H.P. at 240 r.p.m. The propeller weighing 10000 lb-wt. is carried by the shaft overhanging the support by 15″, propeller thrust is 30000 lb-wt., calculate the maximum direct stress induced in the cross-section of the propeller shaft. (9825 lb/in^2.)

(12) A shaft, 10 feet long, 3 inches in diameter, is subjected to a twisting moment of 18000 lb-inches and carries in the centre a pulley weighing 500 lb. Assuming that the shaft is freely supported at the ends, determine the maximum intensity of shear stress. Neglect the weight of the shaft. (L. U.)

(4420 lb/in^2.)

(13) A hollow shaft is subjected to a torque of 200,000 lb-feet and a bending moment of 100,000 lb-feet. The internal diameter of the shaft is one-half the external diameter. If the maximum shear stress is not to exceed 5 tons/in^2., find the diameter of the shaft. (L. U.)

(10·91″.)

(14) A waggon weighing 2 tons is moving at $1\frac{1}{2}$ m.p.h. How many springs, each of 24 coils, will be required in a buffer-stop to absorb the energy of motion during a compression of 8″? The mean diameter of coils is 9″, the diameter of the steel rod comprising the coils is $\frac{3}{4}$″. $N = 12 \times 10^6$ lb/in^2. (5.)

(15) A closely coiled helical spring is made of steel wire $\frac{1}{2}''$ diameter. The number of coil is 20 and the mean radius of each coil is $3''$. Calculate the stiffness of the spring. $N = 12 \times 10^6$ lb/in^2.
(21·7 lb. per inch.)

(16) A spiral spring made of $\frac{1}{4}''$ diameter wire, has 20 close coils, each $3''$ mean diameter. Find the axial load the spring will carry if the stress is not to exceed 12 tons/in^2. Also determine the extension of the spring and the work done per unit volume of the spring. $N=5000$ tons/in^2. (L. U.)
(55 lb.; $5{\cdot}431''$; 16·13 inch-lbs.)

(17) A safety valve of $3''$ diameter is to blow off at a pressure of 150 lb/in^2. by gauge. It is held by a close-coiled compression spring of circular steel bar. The mean diameter is $6''$ and the initial compression is $1''$. Find the diameter of the steel bar and the number of convolutions necessary if the shear stress allowed is 8 tons/in^2. $C=5000$ tons/in^2. (C. U.)
($\cdot967''$ say, $1''$; 6·1 turns.)

(18) A helical spring is made of steel of square section $\frac{1}{4}''$ side with 18 coils of $3''$ mean diameter. Taking $N = 12 \times 10^6$ lb/in^2., find the deflection under an axial load of 25 lb. and the shear stress induced. ($1{\cdot}45''$; 11538 lb/in^2.)

(19) A weight of 30 lb. is dropped on to a helical compression spring made of $\frac{1}{2}''$ steel wire closely coiled to a mean diameter of $6''$ with 20 coils. If the instantaneous compression is $5''$, calculate the height of drop. $N = 12 \times 10^6$ lb/in^2. ($4{\cdot}04''$.)

(20) A weight of 50 lb. is dropped through a height of $30''$ on to a closely coiled compression spring which instantaneously compresses by $6''$ under the impact. If the diameter of the steel rod is $1''$ and the mean radius of the coils is $4''$, find the maximum instantaneous stress produced by the impact and the number of coils in the spring. $N = 12 \times 10^6$ lb/in^2. (12220 lb/in^2.; 29·29, say 30 turns.)

(21) An open coil helical spring is made of $\frac{1}{2}''$ steel rod coiled to a mean radius of $4''$. If the number of coils is 12 and the angle of the helix is 30°, calculate the deflection under an axial load of 20 lb. $E = 30 \times 10^6$ lb/in^2.; $N = 12 \times 10^6$ lb/in^2. ($1{\cdot}438''$.)

(22) A steel shaft *ABCD* of circular section is $72''$ long and is supported in bearings at the ends A and D. $AB = 27''$; $BC = 18''$ and $CD = 27''$. The shaft is horizontal and two horizontal arms, rigidly connected to the shaft at B and C, project from it at right angles on opposite sides. The arm at B carries a vertical load of 4000 lb. at $12''$ from the shaft axis and the arm at C carries a vertical balancing load at $15''$ from the axis. If the shearing stress is not to

exceed 5 tons/in^2., determine the minimum permissible diameter for the shaft. Assume that the bearings give simple point support to the shaft. (L. U.)

(3·695″, say, 4″.)

(23) A steel shaft *ABCD* has a total length of 51″ made up as follows: $AB = 12''$; $BC = 15''$ and $CD = 24''$. *AB* is hollow, its outside diameter being 4″ and its inside diameter *d* inches. *BC* and *CD* are solid, having diameters of 4″ and 3·5″ respectively. If equal opposite torques are applied to the ends of the shaft, find to the nearest 0·05″, the maximum permissible value of *d* for the maximum shearing stress in *AB* not to exceed that in *CD*.

If the torque applied to the shaft is 36 ton-inches, what is the total angle of twist? $C = 5300$ tons/in^2. (L. U.)

($d = 3''$; 1·138°.)

(24) A hollow steel shaft has an external diameter of 9″ and it was found that when running at 125 r.p.m. and transmitting 2950 H.P., the angle of twist per foot length of shaft was 0·15 degree. If C is assumed to be 12×10^6 lb/in^2., estimate the internal diameter of the shaft and the maximum shearing stress.

If the total strain-energy U is expressed as $U = u \times$ volume of shaft, find the value of u; find also the strain-energy per cubic inch of material at the external surface of the shaft. (L. U.)

(5·283″; 11780 lb/in^2; 3·886 inch-lb; 5·781 inch-lb.)

(25) A solid steel shaft $2\frac{1}{4}''$ diameter is fixed rigidly and co-axially inside a bronze sleeve $3\frac{1}{2}''$ external diameter. Calculate the angle of twist in a 5 foot length of the composite shaft due to the action of a pure torque of 3·75 tons-inches. Take C for steel as 5100 tons/in^2., and for bronze 2700 tons/in^2. (L. U.)

(0·2816 degree.)

(26) A tube with closed ends has an external diameter of 1·5″ and the thickness of the metal is 0·04″. It is subjected simultaneously to an internal pressure of 0·5 ton/in^2. and an axial torque of 1·5 ton-inches. Find the magnitude and directions of the principal stresses at the outer surface of the tube. Show on a sketch the directions of the principal planes and principal stresses relative to the axis of the tube.

If the yield point of the metal in simple shear is 12 tons/in^2., and the maximum shearing stress is assumed to be the criterion of failure, find the pressure at which failure will occur, the torque being kept constant at 1·5 ton-inches. (L. U.)

$$\left\{ \begin{array}{l} p_1 = 9{\cdot}156 \text{ tons/in}^2\text{., tensile; } p_2 = 4{\cdot}156 \text{ tons/in}^2\text{., tensile;} \\ \theta_1 = 13^\circ\ 43'; \ \theta_2 = 76^\circ\ 17'; \text{ pressure, } 2{\cdot}547 \text{ tons/in}^2. \end{array} \right\}$$

(27) A closely coiled helical spring made from round rod is required to carry a load of 180 lb. for a maximum stress not exceeding 60000 lb/in^2., the spring stiffness being 45 lb. per inch. The diameter of the helix is 3·5″ and the modulus of rigidity for the material 12×10^6 lb/in^2. Calculate (a) the diameter of the rod to the nearest $\frac{1}{16}$″; (b) the number of turns required in the spring. (L. U.)

$$\left\{\frac{5''}{16};\ 7{\cdot}42 \text{ turns.}\right\}$$

(28) A closely coiled helical spring of round section steel wire is required to have a stiffness of 45 lb. per inch and when carrying its maximum load of 60 lb. the shearing stress is not to exceed 50000 lb/in^2. The spring is to fit freely in a recess $1\frac{3}{4}$″ diameter. Find a suitable diameter for the wire from the gauge sizes given and also the mean diameter of the coil and number of turns.

Gauge No.:	6	7	8	9	10
Diam. of wire in inch:	0·192	0·176	0·160	0·144	0·128

Take the modulus of rigidity as 5250 tons/in^2.

(L. U.)

($d = 0{\cdot}176''$; $D = 1{\cdot}5''$; $n = 9{\cdot}285$.)

(29) A helical spring has a mean diameter of $2\frac{1}{2}$″ and consists of 12 turns of 0·25″ diameter wire. Initially the inclination of the turns is 7°. Calculate the axial and angular deformation caused by an axial "winding up" couple of 80 lb-inches.
$E = 30 \times 10^6$ lb/in^2.; $C = 12 \times 10^6$ lb/in^2.

(L. U.)

($\delta = 0{\cdot}04992''$; $\phi = 1{\cdot}32$ radians.)

(30) An open coiled spring of 5″ mean diameter has 10 coils of $\frac{1}{2}$″ diameter wire, at a slope of 30° to the horizontal, when the coil axis is vertical. Find expressions for the longitudinal extension δ and the rotation θ for the joint application of an axial load W and an axial torque T.

Hence find the axial load and torque necessary to extend the spring 0·2″ if rotation is prevented, indicating whether the torque tends to wind or unwind the spring.
$E = 30 \times 10^6$ lb/in^2.; $N = 12 \times 10^6$ lb/in^2. (L. U.)

($W = 13{\cdot}8$ lb-wt.; $T = -3{\cdot}52$ lb-inches.)

CHAPTER XIII

ELEMENTS OF REINFORCED CONCRETE

1. Reinforced concrete beams: In the study of beams, we have seen that a portion of the beam-section is necessarily under tension. Concrete, as a material, can resist compression but it cannot be relied on to take tension. If concrete has to be used as a material for beams, it must be reinforced by a suitable material which will take tension, so that the two together can form a composite section to resist the compression and tension due to the bending moment at the section. Concrete and steel form an ideal combination for a composite beam section, owing to the happy circumstance that concrete contracts in setting and so, grips the steel reinforcement. The combination is ideal for another reason, viz., that both materials have about the same coefficient of expansion. If the coefficient had been different, with a change in temperature, the steel bars comprising the reinforcement would tend to work loose and slip past the enclosing concrete in which case, the condition of the composite section, viz., that the concrete takes the compression while the steel takes care of the tension, would cease to operate, each material behaving separately as a beam. In the following articles, the concrete is supposed to be of 1:2:4 mix., i.e., 4 parts by volume of aggregate, 2 parts of sand and one part of cement. For ordinary grade concrete, the ultimate crushing stress at 28 days' tests, is of the order of 3000 to 4000 lb/in^2. The working stress is therefore taken at 600 to 1000 lb/in^2. The mild steel used in reinforced concrete has the usual strength viz., an ultimate stress of 28 to 33 tons/in^2., so that the working stress is taken at 16000 to 18000 lb/in^2.

2. Assumptions made in the simple theory: In the theory for stresses set up in a reinforced concrete beam-section which we shall discuss, certain fundamental assumptions are made, which may be stated as under:

(i) Sections which are plane before bending remain plane after bending.

(ii) Concrete is effective only for the purpose of offering compressive stresses. No tension is taken by concrete. In other words, the tension on one side of the neutral axis of the beam-section will be resisted entirely by the steel reinforcement.

(iii) The modulus of Elasticity for concrete bears a constant ratio with that for steel. If E_s and E_c are respectively the moduli for steel and concrete, the modular ratio $\frac{E_s}{E_c} = m$, a constant.

3. **Singly reinforced rectangular beam:** Consider a beam of rectangular section of width b and carrying reinforcement of sectional area A_t at a distance of d from the top edge;

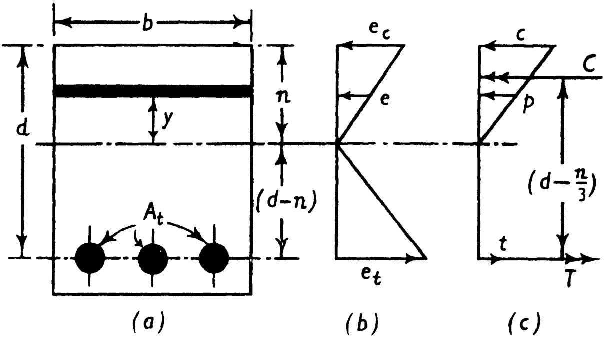


FIG. 194

d is called the "effective" depth of the beam, the portion of concrete below the bars forming a cover.

The position of the neutral axis:

Let n be the distance of the N.A. from the top edge. Consider an elementary strip of the section δa at a height y above the N.A. From assumption (i) The strain $e = \frac{y}{R}$, i.e., the strain is proportional to the distance from the neutral axis. The maximum compressive strain $e_c = \frac{n}{R}$ will occur at the top edge. Below the neutral axis, the strain follows the same straight line law, the value of the strain e_t at the centre of the reinforcing bars being given by,

$$e_t = \frac{(d-n)}{R}.$$

The variation of the strain is shown in fig. 194(*b*). If the elastic limit is not exceeded, stress is proportional to strain. Therefore, $p = E_c \times e$. The variation of stress is a straight line variation as shown in fig. 194(*c*). If c is the maximum intensity of compressive stress,

$$c = E_c \times e_c.$$

If t is the intensity of tensile stress induced in the steel-rods,

$$t = E_s \times e_t.$$

We have, therefore,

$$\frac{e_c}{e_t} = \frac{n}{d-n} = \frac{E_s}{E_c}\cdot\frac{c}{t} = \frac{mc}{t}$$

We get the relation,

$$\frac{mc}{t} = \frac{n}{d-n} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

The elementary thrust δC on the elementary strip δa is $p.\delta a$. The total thrust C in the compression area,

$= \Sigma\delta C =$ average intensity of stress $\times$ compression area, since the section is rectangular.

$= \frac{1}{2}c.b.n.$

The total pull T in the reinforcing rods,

$= t.A_t.$

Since $C = T$, we have,

$$\tfrac{1}{2}c.b.n = t\,A_t \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Between equations (1) and (2), eliminate $\frac{c}{t}$ and solve for n which gives the position of the N.A.

We have, $\dfrac{n}{d\ \ n} = \dfrac{2mA_t}{bn}.$

Putting $p = \frac{A_t}{bd}$, the ratio of the steel sectional area and the concrete section, we get,

$$\frac{n}{d-n} = \frac{2mp.b.d}{bn}$$

$$\therefore \quad n^2 + 2mp.nd = 2mp.d^2$$

$$\therefore \quad \left(\frac{n}{d}\right)^2 + 2mp\left(\frac{n}{d}\right) = 2mp.$$

Putting $\frac{n}{d} = n_1$, the equation simplifies to,

$$n_1^2 + 2mp.n_1 = 2mp$$

$$\therefore \quad (n_1 + mp)^2 = m^2p^2 + 2mp$$

$$\therefore \quad n_1 = \frac{n}{d} = \sqrt{m^2p^2 + 2mp} - mp \quad \ldots\ldots\ldots\ldots\ldots (3)$$

Moment of Resistance:

The moment of resistance $M_r = C$ or $T \times$ lever arm a of the couple. The total thrust C acts at the *c.g.* of the compression triangle of stress, i.e., at $\frac{n}{3}$ from the top. The total pull T acts at the centre of the reinforcing rods. The lever arm a is, therefore, given by, $a = d - \frac{n}{3}$.

Since we know the value of n for a given section, the value of a is also known:

$$\therefore \quad M_r = \tfrac{1}{2}\,c.b.n.a = tA_t.a, \text{ where } a = \left(d - \frac{n}{3}\right).$$

Given the cross-section of a beam with its reinforcement, the values of the stresses set up in concrete and steel to resist a given bending moment can now be worked out.

Example 1.

A reinforced concrete beam 10″ wide and 20″ deep has four 1″ diameter reinforcing bars placed with their centres 2″ above the bottom side of the beam. At a certain cross-section it has to resist a bending moment of 500000 lb-inches. Calculate the stresses induced in the concrete and steel, taking the modular ratio m = 15.

$$A_t = 4 \times \cdot 7854 = 3\cdot 1416 \text{ in}^2.\text{; effective depth} = 20 - 2 = 18''.$$

$$mp = \frac{15 \times 3\cdot 1416}{10 \times 18} = \cdot 2618$$

$$\therefore \quad 2mp = \cdot 5236$$

$$m^2p^2 = \cdot 0686$$

$$\therefore \quad \sqrt{m^2p^2 + 2mp} = \sqrt{\cdot 5922} = \cdot 7695$$

$$n_1 = \frac{n}{d} = \sqrt{m^2p^2 + 2mp} - mp = \cdot 7695 - \cdot 2618 = \cdot 5077$$

$$\therefore \quad n = \cdot 5077 \times 18 = 9\cdot 1386''.$$

$$a = d \quad \frac{n}{3} = 18 - 3\cdot 0462 = 14\cdot 954''.$$

$$M_r = \tfrac{1}{2}\, c.b.n.a = \tfrac{1}{2}\, c \times 10 \times 9\cdot 14 \times 14\cdot 95 = 500000$$

$$\therefore \quad c = 732 \text{ lb/in}^2.$$

Also $\dfrac{mc}{t} = \dfrac{n}{d - n}$

$$\therefore \quad \frac{15 \times 732}{t} = \frac{9\cdot 14}{8\cdot 86}$$

$$\therefore \quad t = 10640 \text{ lb/in}^2.$$

Example 2.

A reinforced concrete beam 8″ wide and 18″ deep including 2″ cover has three reinforcing bars $\frac{7}{8}''$ in diameter. Find the position of the neutral axis and the safe moment of resistance if c and t are not to exceed 750 and 18000 lb/in²., respectively. Take m = 18. What uniformly distributed load will the beam carry over a span of 16 feet?

$$A_t = 3 \times \cdot 6 = 1\cdot 8 \text{ in}^2.\text{; effective depth} = 16''.$$

$$mp = \frac{18 \times 1\cdot 8}{8 \times 16} = \cdot 2531$$

$$\therefore \quad 2mp = \cdot 5062$$

$$m^2p^2 = \cdot 0641$$

$$\therefore \quad m^2p^2 + 2mp = \cdot 5703$$

$$\therefore \quad \sqrt{m^2p^2 + 2mp} = \cdot 7552$$

$$n_1 = \frac{n}{d} = \sqrt{m^2p^2 + 2mp} - mp = \cdot 7552 - \cdot 2531 = \cdot 5021$$

$$\therefore \quad n = \cdot5021 \times 16 = 8\cdot034''.$$

$$a = d \quad \frac{n}{3} = 16 \quad 2\cdot68 = 13\cdot32''.$$

Since $\dfrac{mc}{t} = \dfrac{n}{d - n}$, when $c = 750$ lb/in^2., the corresponding t will be given by, $\dfrac{18 \times 750}{t} = \dfrac{8\cdot034}{7\cdot966}$

$\therefore \quad t = \dfrac{7\cdot966}{8\cdot034} \times 13500$ or 13390 lb/in^2., which being less than 18000 lb/in^2., is permissible.

The moment of resistance is,

$$M_r = t\,A_t.a = 13390 \times 1\cdot8 \times 13\cdot32 = 321000 \text{ lb/inches.}$$

If the beam is freely supported over a span of 16 feet, carrying a load W uniformly distributed over it, the central moment is,

$$M_c = \frac{Wl}{8} \times 12 \text{ lb-in.} = \frac{W \times 16}{8} \times 12 = 24W \text{ lb-inches.}$$

Equating the bending moment to the moment of resistance,

$$24W = 321000$$

$\therefore \quad W = 13375$ lb-wt. or 836 lb. per foot run.

Example 3.

A reinforced concrete beam 12" wide and 22" deep including a two-inch cover has to carry a uniformly distributed load of 1500 lb. per foot run including its own weight over a span of 20 feet. What area of steel reinforcement must be provided if the maximum stress in concrete is not to exceed 700 lb/in^2.? What will be the corresponding stress in steel? Take $m = 15$.

The maximum bending moment to be resisted will be,

$$\frac{Wl}{8} \times 12 = \frac{30000 \times 20}{8} \times 12 = 900000 \text{ lb-inches.}$$

The moment of resistance,

$$M_r = \tfrac{1}{2}c.b.n. \left(d \quad \frac{n}{3}\right) = \tfrac{1}{2} \times 700 \times 12 \times n \left(20 - \frac{n}{3}\right)$$

Equating,

$$4200n\ (20 \quad \frac{n}{3}) - 900000$$

$$\therefore \quad n(60-n) = 643$$

$$\therefore \quad n^2 - 60n + 643 = 0$$

$$\therefore \quad (n-30)^2 = 900 - 643 = 257$$

$$\therefore \quad n - 30 = -16{\cdot}03$$

$$\text{or} \quad n = 13{\cdot}97'', \text{ say, } 14''.$$

Since $\frac{mc}{t} = \frac{n}{d-n}$, when $c = 700$ lb/in^2, the corresponding t will be given by,

$$\frac{15 \times 700}{t} = \frac{14}{6}$$

$$\therefore \quad t = 4500 \text{ lb/in}^2.$$

Also, since $C = T$,

$$\tfrac{1}{2}c.b.n = t.A_t$$

$$\therefore \quad \tfrac{1}{2} \times 700 \times 12 \times 14 = 4500.A_t$$

$$\therefore \quad A_t = \frac{14 \times 14}{15} = 13 \text{ in}^2.$$

Economic percentage of reinforcement: The working stress for mild steel is about 16000 to 18000 lb/in^2. In the above example, when concrete is stressed to 700 lb/in^2., the corresponding stress in the steel reinforcement is only 4500 lb/in^2. As a design, therefore, the section is very wasteful, as the steel, which is capable of offering a stress of 16000 to 18000 lb/in^2., is called upon to offer only 4500 lb/in^2. and as many as 13 sq. inches have to be provided for the steel sectional area.

In an economical design, the concrete section and the steel reinforcement provided should be so proportioned that both the materials should reach the maximum permissible stresses when the given bending moment comes upon the section.

We shall now work out these proportions for a singly reinforced beam of rectangular section, for given values of the permissible stresses and a given modular ratio.

(i) $c = 1000$ lb/in^2.; $t = 18000$ lb/in^2.; $m = 15$.

$$\frac{mc}{t} = \frac{n}{d-n}$$

Substituting the limits for c and t,

$$\frac{n}{d-n} = \frac{15 \times 1000}{18000} = \tfrac{5}{6}$$

$\therefore \quad n = \tfrac{5}{11}d = \cdot455d$

The lever arm $a = d - \frac{n}{3} = d\,(1 - \tfrac{5}{33}) = \frac{28d}{33}$

$= \cdot848d.$

Since $\tfrac{1}{2}c.b.n = t.A_t = t.p.b.d$

$\tfrac{1}{2} \times 1000 \times \tfrac{5}{11} = 18000.p$

$\therefore \quad p = \tfrac{5}{396} = 1{\cdot}26\%.$

The moment of resistance is,

$$M_r = t.A_t.a = 18000 \times \frac{1{\cdot}26}{100} \times bd \times \frac{28}{33}d = 193bd^2.$$

(ii) $c = 750$ lb/in^2.; $t = 18000$ lb/in^2.; $m = 15$.

$$\frac{mc}{t} = \frac{n}{d-n}$$

Substituting the limits for c and t,

$$\frac{n}{d-n} = \frac{15 \times 750}{18000} = \tfrac{5}{8}$$

$\therefore \quad \frac{n}{d} = \tfrac{5}{13} \quad \therefore\ n = \tfrac{5}{13}d = \cdot385d$

The lever arm $a = d - \frac{n}{3} = d - \frac{5d}{39} = \frac{34d}{39} = \cdot872d$

Since $\tfrac{1}{2}c.b.n = t.A_t = t.p.b.d$, we have,

$\tfrac{1}{2} \times 750 \times \tfrac{5}{13} = 18000 \times p$

$$\therefore \quad p = \frac{375 \times 5}{13 \times 18000} = \frac{\cdot8}{100}$$

$$\therefore \quad p = \frac{A_t}{bd} = \cdot8\%.$$

The moment of resistance is,

$$M_r = t.A_t.a = 18000 \times \frac{\cdot 8}{100} \times bd \times \frac{34}{39}.d$$

$$= 126bd^2.$$

(iii) $c = 750$ lb/in².; $t = 18000$ lb/in².; $m = 18$.

$$mc \qquad\quad n$$
$$t \quad - \quad d \quad n$$

Substituting,

$$n \qquad 18 \times 750 \qquad 3$$
$$d - n \qquad 18000 \qquad 4$$

$$\therefore \quad n \quad \frac{3d}{7} \quad \cdot 43d$$

$$a \quad d \quad \frac{n}{3} \quad d \quad \frac{d}{7} - \frac{6d}{7} - \cdot 86d.$$

$\frac{1}{2}c.b.n = t\ A_t = t.p.b.d$

$\therefore \quad \frac{1}{2} \quad 750 \quad \frac{3}{7} - \quad 18000p$

$$\therefore \quad p = \frac{5}{560} = \cdot 893\%$$

$$M_r = t.A_t.a = 18000 \times \frac{\cdot 893}{100} \times bd \times \tfrac{6}{7}d = 138bd^2.$$

(iv) $c = 600$ lb/in².; $t = 16000$ lb/in².; $m = 15$.

$$\frac{mc}{t} = \frac{n}{d-n}$$

Substituting,

$$n \qquad 15 \times 600 \qquad 9$$
$$d \quad n \; - \; 16000 \qquad 16 \cdot$$

$$\therefore \quad n = \frac{9}{25}d = \cdot 36d.$$

$$a = d - \frac{n}{3} = d - \cdot 12d = \cdot 88d$$

$\frac{1}{2}c.b.n = t.A_t = t.p.b.d$

$$\therefore \quad \tfrac{1}{2} \times 600 \times \frac{9}{25} = 16000 \times p$$

$$\therefore \qquad p \doteq \frac{108}{16000} = \frac{\cdot 675}{100} \text{ i.e., } \cdot 675\%$$

$$M_r = t.A_t.a = 16000 \times \frac{\cdot 675}{100} \times bd \times \cdot 88d$$

$$= 95 \cdot 04\ bd^2, \text{ say, } 95bd^2.$$

Example 4.

For the beam in example 3, calculate the effective depth and the necessary reinforcement, keeping the width as 12″. The permissible stresses are c = 700 lb/in².; t = 18000 lb/in².; m = 15.

$$\frac{mc}{t} = \frac{n}{d-n}$$

$$\therefore \qquad \frac{n}{d-n} = \frac{15 \times 700}{18000} = \frac{7}{12}$$

$$\therefore \qquad n = \tfrac{7}{19}d = \cdot 37d$$

$$a = d - \frac{n}{3} = \cdot 877d.$$

Since, $\tfrac{1}{2}c.b.n = t.A_t = t.p.bd$

$$\tfrac{1}{2} \times 700 \times \tfrac{7}{19} = 18000p$$

$$\therefore \qquad p = \cdot 7163\%.$$

$$M_r = \tfrac{1}{2}c.b.n.a = \tfrac{1}{2} \times 700 \times b \times \tfrac{7}{19}d \times \cdot 877d$$

$$= 113 \cdot 1\ bd^2.$$

Equating the bending moment to the moment of resistance,

$$113 \cdot 1\ bd^2 = 900000.$$

Since $b = 12''$,

$$d^2 = \frac{900000}{113 \cdot 1 \times 12}$$

$$\therefore \qquad d = 25 \cdot 75''$$

$$A_t = \frac{\cdot 7163}{100} \times 12 \times 25 \cdot 75 = 2 \cdot 214 \text{ sq. inches.}$$

Provide 4 numbers $\frac{7}{8}''$ diameter bars.

Example 5.

A reinforced beam of rectangular section, span 20 feet, carries a uniformly distributed load of 500 lb. per foot run, inclusive of its own *weight. If the permissible values of c and t are 750 and 18000 lb/in².,* *respectively and m = 15, design a suitable section.*

$$\text{Maximum B.M.} = \frac{Wl}{8} \times 12 = \frac{500 \times 20 \times 20}{8} \times 12$$
$$= 300000 \text{ lb-inches.}$$

The moment of resistance for an economical section with the given stresses and the modular ratio of 15 is $126bd^2$.

Equating, $126bd^2 = 300000$.

If we make the effective depth d twice the width b, we have,

$$63d^3 = 300000$$
$$\therefore \quad d = 16{\cdot}82''$$
$$\text{and} \quad b = 8{\cdot}41''.$$

The necessary reinforcement A_t is ·8% of the concrete section.

$$\therefore \quad A_t = \frac{{\cdot}8}{100} \times 8{\cdot}41 \times 16{\cdot}82 = 1{\cdot}132 \text{ in}^2.$$

A beam 9″ wide and 18″ deep including $1\frac{1}{2}''$ cover with four $\frac{5}{8}''$ diameter rods for reinforcement will do.

5. Design of slabs: In the ordinary beam and slab construction, the slab can be dealt with as a beam of rectangular section, while the beam consists of the web below the slab and a portion of the slab so that it is a beam of *T*-section. This section will be studied later.

Consider a slab freely supported over a span l. It is sufficient to consider a foot width of the slab so that $b = 12''$. If W is the uniformly distributed load on the slab, the maximum bending moment $M = \frac{Wl}{8}$. The necessary effective depth for the slab and the reinforcement can be calculated on the basis of an economic rectangular section.

If the permissible stresses are $c = 750$ lb/in²., and $t = 18000$ lb/in²., with a modular ratio of 15,

$$M_r = 126bd^2.$$

Equating it to the bending moment M, we have,

$$126bd^2 = M$$

$$\therefore \quad d^2 = \frac{M}{126b} = \frac{M}{1512}, \text{ since } b = 12''.$$

$$\therefore \quad d = \sqrt{\frac{M}{1512}}.$$

The reinforcement will be ·8% of thc concrete section.

If we take the permissible stresses as 600 lb/in^2., and 16000 lb/in^2. for concrete and steel with $m = 15$,

$$M_r = 95bd^2$$

$$\therefore \quad d^2 = \frac{M}{95b} = \frac{M}{1140}, \text{ since } d = 12''$$

$$\therefore \quad d = \sqrt{\frac{M}{1140}}.$$

The reinforcement will be ·675% of the concrete section.

Example 6.

Design a slab, freely supported over a span of 10 feet, to carry a floor load of 180 lb. per sq. foot, inclusive of its own weight. The permissible stresses are $c = 750$ lb/in^2., $t = 18000$ lb/in^2. and $m = 15$.

Taking a foot width of slab,

$$M = \frac{Wl}{8} \times 12 = \frac{180 \times 1 \times 10 \times 10}{8} \times 12 = 27000 \text{ lb-inches.}$$

Equating,

$$126bd^2 = 27000$$

$$\therefore \quad d^2 = \frac{27000}{1512} = 17{\cdot}85$$

$\therefore$ effective depth $d = 4{\cdot}226''$, say, $4\frac{1}{4}''$.

With a $\frac{3}{4}''$ cover to protect the reinforcement the total depth of slab will be $5''$.

Per foot width of slab,

$$A_t = \frac{\cdot 8}{100} \times 12 \times 4{\cdot}25 = \cdot 408 \text{ in}^2.$$

Using $\frac{3}{8}''$ rods, spacing centre to centre will be given by,

$$\frac{\cdot 11 \times 12}{\cdot 408} = 3{\cdot}24'', \text{ say, } 3''.$$

Using $\frac{1}{2}''$ rods, the spacing will be,

$$\frac{\cdot 1963 \times 12}{\cdot 408} = 5{\cdot}8'', \text{ say, } 5\tfrac{1}{2}'' \text{ centre to centre.}$$

Example 7.

A floor slab is continuous over beams spaced 12 feet apart centre to centre. It has to carry a load 150 lb. per sq. foot, exclusive of its own weight. Using the same stresses and modular ratio as in example 6, calculate the thickness of slab and the reinforcement required.

Per foot width of slab,

Live load = 150 × 12 × 1 = 1800 lb-wt.

Dead load = 12 × 6 × 12 = 864 lb-wt., assuming the slab to be 6″ thick and taking the density of concrete as 144 lb. per cubic foot.

Total load W = 2664 lb-wt.

$$\text{Maximum bending moment } M = \frac{Wl}{10} \times 12 = \frac{2664 \times 12}{10} \times 12 = 38370 \text{ lb-inches.}$$

$$126bd^2 = 38370$$

$$\therefore \quad d^2 = \frac{38370}{1512} = 25{\cdot}37$$

$$\therefore \quad d = 5{\cdot}037.$$

With about an inch cover, the slab will be 6″ thick.

Per foot width of slab,

$$A_t = \frac{\cdot 8}{100} \times 12 \times 5{\cdot}037 \text{ in}^2. = \cdot 4837 \text{ in}^2.$$

Using $\frac{1}{2}''$ diameter rods, the spacing will be,

$$\frac{\cdot 1963 \times 12}{\cdot 4837} = 4\cdot 871'', \text{ say, } 4\tfrac{1}{2}'' \text{ centre to centre.}$$

6. Short columns under axial loading: Since concrete and steel are resisting the load jointly, the strain in the two materials will be the same. If c is the stress in concrete, the corresponding stress in steel will be $m.c.$ If A is the area of column section and A_s that of the steel reinforcement, the safe axial load will be,

$$\begin{aligned} P &= c\ (A - A_s) + mc.A_s \\ &= c\,\{A + (m - 1)\ A_s\}. \end{aligned}$$

This formula has been used in Chapter I for solving example 10.

The Code of Practice, however on practical considerations, specifies the following simple formula:

$P = f_c A_c + f_s A_s$, where $f_c = 760$ lb/in^2., for concrete of 1:2:4 mix. $f_s = 18000$ lb/in^2., for steel, $A_c =$ the cross-sectional area of concrete, not including any finishing material applied after the column is cast and $A_s =$ the sectional area of compression reinforcement. In countries where the standards of material and workmanship are steadily improving, the stresses specified in the formula are allright. It is safer however, to use values of $f_c = 600$ lb/in^2. and $f_s = 13500$ lb/in^2.

The reinforcement, A_s should not be less than ·8% nor more than 8% of the column area. Depending upon the size of the column, the main reinforcing bars may vary from $\frac{1}{2}''$ to $1\frac{1}{2}''$ in diameter. The minimum cover of concrete for the bars should be $1''$ or the diameter of the bar, whichever is greater.

At suitable intervals, the longitudinal reinforcing bars must be provided with binders or stirrups to prevent buckling. These vary in size from $\frac{1}{4}''$ to $\frac{1}{2}''$ diameter and should be spaced at not less than $6''$ nor more than $12''$ centre to centre.

Detailed study of columns and footings will be taken up later.

CHAPTER XIV

STRESSES IN FRAMES: STATIC LOADING

1. Method of sections: Stresses in perfect frames under static loading can be obtained graphically by drawing the usual stress diagrams known as Maxwell's diagrams. We shall obtain the pulls and pushes in the members of a loaded truss, by analysis, using the Method of Sections to evaluate them.

There is nothing special about the method of sections, which is only an application of the usual statical laws of equilibrium to a portion of a framed structure. Consider, for example, the French truss shown in fig. 195.

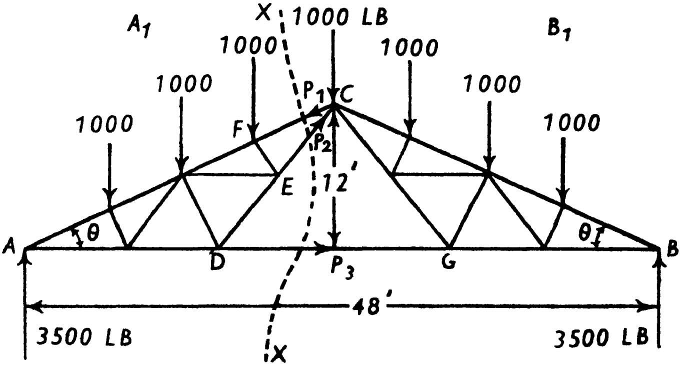


FIG. 195

Let the section XX, cutting the numbers FC, EC and DG, divide the truss into two portions A_1 and B_1. Since the truss is in equilibrium, each portion A_1 or B_1 is in equilibrium under the system of forces acting on it. For the portion A_1, for instance, the system consists of the upward reaction of 3500 lb-wt. at the support A, the three vertical loads of 1000 lb-wt. each at the joints and the pulls pushes P_1, P_2 and P_3 exerted by B_1 on A_1 at the sections where XX cuts the truss. These seven forces keep the portion A_1 of the truss in equilibrium. The algebraic sum of the resolved parts of these forces in any direction must be zero; likewise, the algebraic sum of their moments about any point must be zero. To evaluate P_3, the pull in the tie *DG*, let us take moments of the forces about the apex *C*. We have,

$$P_3 \times 12 = 3500 \times 24 - 1000\ (6 + 12 + 18)$$

$$\therefore \quad P_3 = 7000 - 3000 = 4000 \text{ lb-wt. tensile.}$$

To evaluate P_2, the pull in *EC*, we take moments about *A*. Since $\angle CAD = \angle ACD$, we have,

$$P_2.\ AC.\sin\theta = 1000\ (6 + 12 + 18)$$

$$\therefore \quad P_2 \times 24 \sec\theta.\ \sin\theta = 36000$$

$$\therefore \quad P_2 \tan\theta = 1500$$

$$\text{or} \quad P_2 = 3000 \text{ lb-wt. tensile, since } \tan\theta = \tfrac{1}{2}.$$

Similarly to obtain P_1, the thrust in *FC*, take moments about the joint *D*.

$$\text{Now} \quad AC = 24 \sec\theta$$

$$\text{and} \quad AD = \tfrac{1}{2} AC.\ \sec\theta = 12 \sec^2\theta = 12\ (1 + \tan^2\theta)$$

$$= 12\ (1 + \tfrac{1}{4}) = 15 \text{ feet.}$$

Taking moments about *D*,

$$\begin{aligned} P_1 \times 15\sin\theta &= 3500 \times 15 - 1000\ (15 - 6) - 1000(15 - 12) \\ &\quad + 1000\ (18 - 15) \\ &= 52500 - 9000 - 3000 + 3000 \\ &= 43500 \end{aligned}$$

$$\therefore \quad P_1 \times 15 \times \frac{1}{\sqrt{5}} = 43500$$

$$\therefore \quad P_1 = 2900 \sqrt{5} = 6500 \text{ lb-wt., compressive.}$$

It will be noticed that the effectiveness of the method consists in selecting the section *XX* in such a manner that it divides the truss into two portions A_1 and B_1 by cutting only three members of the truss. Then, to evaluate the pull or push in any one of these members, we take moments about the intersection of the other two so that the pulls or pushes across them do not figure in the equation of moments, leaving only one unknown to be solved.

We shall now apply this method to a few trusses under steady loads.

2. Pratt truss: Span 120 feet; eight panels of 15 feet each; height of truss 20 feet; unit loading at the lower joints.

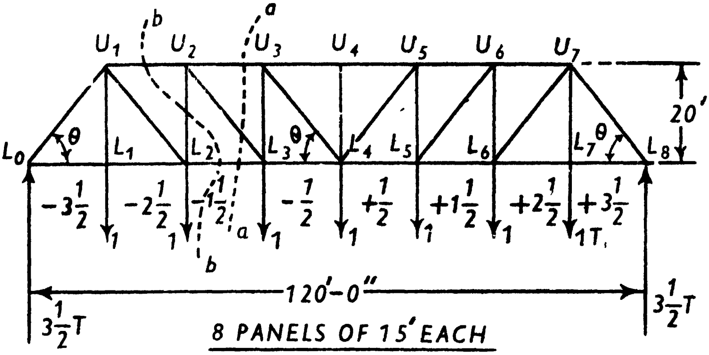


Fig. 196

There is a unit load—say one ton—at each lower joint. The reactions at the supports will be $3\frac{1}{2}$ tons each. We shall work out the stresses in the members of the panel L_2L_3. Let the section *aa* divide the truss into two parts and let us consider the equilibrium of the portion on the left. To evaluate the push in the member U_2U_3, take moments about the joint L_3. We have,

$u_2u_3 \times h = 3\frac{1}{2} \times 3a \quad 1 \times 2a - - 1 \times a = 7{\cdot}5a$, where a is the width of each panel and h is the height of the truss and u_2u_3 is the stress in the member.

$$\therefore \quad u_2u_3 = \frac{7{\cdot}5a}{h} = \frac{7{\cdot}5 \times 15}{20} = 5{\cdot}625 \text{ tons, compressive.}$$

It will be noticed that the expression $7{\cdot}5a$ represents the bending moment due to the load system at the joint L_3. It is evident that the push in the member U_2U_3 is equal to the bending moment at the opposite joint divided by the height of the girder.

Similarly, to obtain the pull in the member L_2L_3, we take moments about the joint U_2. We have,

$$l_2l_3 \times h = 3\tfrac{1}{2} \times 2a - - 1 \times a = 6a = M_{U2}$$

$$\therefore \quad l_2l_3 = \frac{M_{U2}}{h} = \frac{6a}{h} = \frac{6 \times 15}{20} = 4{\cdot}5 \text{ tons, tensile.}$$

For stress in the diagonal U_2L_3, resolve the system of forces vertically. We get,

$u_2l_3 \sin\theta = 3\frac{1}{2} - 1 - 1 =$ shear force in the panel L_2L_3 $= 1\frac{1}{2}$ tons, θ being the inclination of the diagonal to the horizontal.

$\therefore \; u_2l_3 = 1\frac{1}{2} \operatorname{cosec}\theta = 1\frac{1}{2} \times \frac{5}{4} = 1{\cdot}875$ tons, tensile.

Thus, the stress in the diagonal $U_2L_3 = F\operatorname{cosec}\theta$, where F is the shear force in the panel L_2L_3. For stress in the vertical U_2L_2, consider the section *bb*. Resolving vertically,

$u_2l_2 = 3\frac{1}{2} - 1 - 1 =$ shear force in the panel L_2L_3
$= 1\frac{1}{2}$ tons, compressive.

From these type calculations for stresses in the members of the panel L_2L_3, we are now in a position to state that, (i) the stress in any chord member is given by the B.M. at the opposite joint divided by the height of the girder, (ii) the stress in any diagonal $= F\operatorname{cosec}\theta$, where F is the shear force in the panel and (iii) the stress in a vertical $= F$, where F is shear force in the adjoining panel.

All the diagonals except the first and the last are in tension. The first and the last diagonals, as arranged in fig. 196, will be in compression.

The verticals are usually in compression. For the truss shown in fig. 196, the first and the last vertical will be in tension, the pull being equal to the load at the joint. The middle vertical will have no stress.

A framed girder is, after all, a beam, whose sections have to resist the two actions, viz., the bending moment and the shear force. From the above calculation, it will be clear that the chord members resist the bending moment, while the bracing — verticals and diagonals — resists the shear force.

We may complete the calculations as under and calculate the stresses:

$a = 15$ feet; $h = 20$ feet; $\tan\theta = \frac{20}{15} = \frac{4}{3}$

$$M_{U1} = M_{L1} = 3\tfrac{1}{2} \times a = 3{\cdot}5a \qquad \div\, h = 3{\cdot}5 \times \frac{a}{h} = 3{\cdot}5 \times \tfrac{3}{4}$$
$$= 2{\cdot}625$$

$$M_{U2} = M_{L2} = 3{\cdot}5 \times 2a - 1 \times a = 6a \quad \div h = 6 \times \tfrac{15}{20} = 4{\cdot}5$$

$$M_{U3} = M_{L3} = 3{\cdot}5 \times 3a - 1 \times a - 1 \times 2a = 7{\cdot}5a$$

$$\div h = 7{\cdot}5 \times \tfrac{15}{20} = 5{\cdot}625$$

$$M_{U4} = M_{L4} = 3{\cdot}5 \times 4a - 1(a + 2a + 3a) = 8a$$

$$\div h = 8 \times \tfrac{15}{20} = 6.$$

The shear forces in the panels have been shown in fig. 196, with the conventional signs.

$3{\cdot}5 \text{ cosec}\theta = 3{\cdot}5 \times \frac{5}{4} = 4{\cdot}375$

$2{\cdot}5 \text{ cosec}\theta = 2{\cdot}5 \times 1{\cdot}25 = 3{\cdot}125$

$1{\cdot}5 \text{ cosec}\theta = 1{\cdot}5 \times 1{\cdot}25 = 1{\cdot}875$

${\cdot}5 \text{ cosec}\theta = {\cdot}5 \times 1{\cdot}25 = {\cdot}625.$

Table of Stresses

Compression + ; tension −

Member	Stress	Member	Stress
U_1U_2	+ 4·5	U_4U_5	+ 6·0
U_2U_3	+ 5·625	U_5U_6	+ 5·625
U_3U_4	+ 6·0	U_6U_7	+ 4·5
L_0L_1	− 2·625	L_4L_5	− 5·625
L_1L_2	2·625	L_5L_6	− 4·5
L_2L_3	4·5	L_6L_7	− 2·625
L_3L_4	- 5·625	L_7L_8	− 2·625
U_1L_2	− 3·125	U_5L_4	− 0·625
U_2L_3	− 1·875	U_6L_5	− 1·875
U_3L_4	− 0·625	U_7L_6	− 3·125
U_1L_0	+ 4·375	U_7L_8	+ 4·375
U_2L_2	+ 1·5	U_5L_5	+ 0·5
U_3L_3	+ 0·5	U_6L_6	+ 1·5
U_1L_1	− 1	U_7L_7	− 1
$U_4L_4 = 0$			

3. Warren girder: Equilateral triangles, six bays of 20 feet each, loaded at the lower joints as shown.

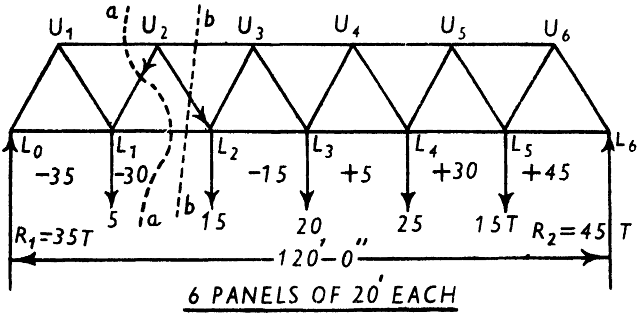


Fig. 197

The width a of each panel $= 20$ feet.

The height h of the girder $= 10\sqrt{3}$ feet.

For the reactions at the supports, taking moments, about L_0,

$$R_2 \times 6a = 5 \times a + 15 \times 2a + 20 \times 3a + 25 \times 4a + 15 \times 5a$$
$$= 270a$$

$\therefore \quad R_2 = 45$ tons

and $\quad R_1 = 35$ tons.

Consider the panel L_1L_2.

Section aa.

Resolving vertically,

$u_2l_1 \sin\theta = 35 - 5 = 30$ tons $=$ shear in the panel.

$$\therefore \quad u_2l_1 = F\operatorname{cosec}\theta = 30 \times \frac{2}{\sqrt{3}} = 20\sqrt{3} \text{ tons, compressive.}$$

Section bb.

Resolving vertically,

$u_2l_2 \sin\theta = 35 - 5 = 30$ tons $=$ shear in the panel.

$$\therefore \quad u_2l_2 = F\operatorname{cosec}\theta = 30 \times \frac{2}{\sqrt{3}} = 20\sqrt{3} \text{ tons, tensile.}$$

The stresses in the diagonals of the panel are $F\operatorname{cosec}\theta$, as before. Only one diagonal is in compression, while the other is in tension.

We may now complete the calculations and tabulate the stresses.

$$M_{U1} = 35 \times \frac{a}{2} = 17{\cdot}5a \qquad \div\, h \quad = \frac{35}{\sqrt{3}}$$

$$M_{L1} = 35 \times a \qquad ,, \quad = \frac{70}{\sqrt{3}}$$

$$M_{U2} = 35 \times \frac{3a}{2} \quad 5 \times \frac{a}{2} = 50a \qquad ,, \quad = \frac{100}{\sqrt{3}}$$

$$M_{L2} = 35 \times 2a - 5 \times a = 65a \qquad ,, \quad = \frac{130}{\sqrt{3}}$$

$$M_{U3} = 35 \times \frac{5a}{2} - 5 \times \frac{3a}{2} - 15 \times \frac{a}{2} = 72{\cdot}5a \qquad ,, \quad = \frac{145}{\sqrt{3}}$$

$$M_{L3} = 35 \times 3a \quad 5 \quad 2a - 15 \times a = 80a \qquad ,, \quad = \frac{160}{\sqrt{3}}$$

$$M_{U4} = 45 \times \frac{5a}{2} \quad 15 \times \frac{3a}{2} - 25 \times \frac{a}{2} = 77{\cdot}5a \qquad ,, \quad = \frac{155}{\sqrt{3}}$$

$$M_{L4} = 45 \times 2a \quad 15 \times a = 75a \qquad ,, \quad = \frac{150}{\sqrt{3}}$$

$$M_{U5} = 45 \times \frac{3a}{2} \quad 15 \times \frac{a}{2} = 60a \qquad ,, \quad = \frac{120}{\sqrt{3}}$$

$$M_{L5} = 45 \times a \qquad ,, \quad = \frac{90}{\sqrt{3}}$$

$$M_{U6} = 45 \times \frac{a}{2} \qquad ,, \quad = \frac{45}{\sqrt{3}}$$

The shear forces in the panels have been marked in fig. 197, with the conventional signs.

$$35 \text{ cosec}\theta = \frac{70}{\sqrt{3}}$$

$$30 \text{ cosec}\theta = \frac{60}{\sqrt{3}}$$

$$15 \text{ cosec}\theta = \frac{30}{\sqrt{3}}$$

$$5 \text{ cosec}\theta = \frac{10}{\sqrt{3}}$$

$$45 \text{ cosec}\theta = \frac{90}{\sqrt{3}}$$

Table of Stresses

Compression + ; tension —

Member	Stress	Member	Stress
U_1U_2	$+70/\sqrt{3}$	L_0L_1	$-35/\sqrt{3}$
U_2U_3	$+130/\sqrt{3}$	L_1L_2	$-100/\sqrt{3}$
U_3U_4	$+160/\sqrt{3}$	L_2L_3	$-145/\sqrt{3}$
U_4U_5	$+150/\sqrt{3}$	L_3L_4	$-155/\sqrt{3}$
U_5U_6	$+90/\sqrt{3}$	L_4L_5	$-120/\sqrt{3}$
		L_5L_6	$-45/\sqrt{3}$
U_1L_1	$-70/\sqrt{3}$	L_0U_1	$+70/\sqrt{3}$
U_2L_2	$-60/\sqrt{3}$	L_1U_2	$+60/\sqrt{3}$
U_3L_3	$-30/\sqrt{3}$	L_2U_3	$+30/\sqrt{3}$
L_3U_4	$-10/\sqrt{3}$	U_4L_4	$+10/\sqrt{3}$
L_4U_5	$-60/\sqrt{3}$	U_5L_5	$+60/\sqrt{3}$
L_5U_6	$-90/\sqrt{3}$	U_6L_6	$+90/\sqrt{3}$

4. Warren girder: Equilateral triangles. Six bays of 20 feet each, loaded at upper and lower joints as shown.

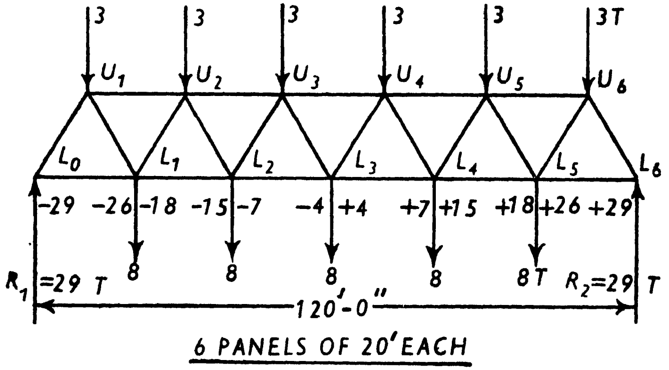


FIG. 198

The only thing to note in a Warren truss loaded at upper and lower joints is that, in any one panel, there are two values of shear, one for the left hand half and the other for the right hand half, the change being due to the load at the upper joint in the panel. For instance, in the panel L_1L_2, the values are 18 tons and 15 tons. For the stresses in the diagonals, therefore, we have,

$l_1u_2 = 18 \operatorname{cosec}\theta = 36/\sqrt{3}$ tons, compressive, and
$l_2u_2 = 15 \operatorname{cosec}\theta = 30/\sqrt{3}$ tons, tensile.

The shear forces in the panels have been noted in fig. 198, with the conventional signs.

The calculations may now be completed and the stresses tabulated, as usual.

$$M_{U_1} = 29 \times \frac{a}{2} = 14{\cdot}5 \times a \qquad \div h = 29/\sqrt{3}$$

$$M_{L1} = 29 \times a - 3 \times \frac{a}{2} = 27{\cdot}5a \qquad ,, = 55/\sqrt{3}$$

$$M_{U2} = 29 \times \frac{3a}{2} \quad 3 \times a - 8 \times \frac{a}{2} = 36{\cdot}5a \qquad ,, = 73/\sqrt{3}$$

$$M_{L2} = 29 \times 2a - 8a - 3\left(\frac{a}{2} + \frac{3a}{2}\right) = 44a \qquad ,, = 88/\sqrt{3}$$

$$M_{U3} = 29 \times \frac{5a}{2} - 8\left(\frac{a}{2} + \frac{3a}{2}\right) - 3(a + 2a) = 47{\cdot}5a \qquad ,, = 95/\sqrt{3}$$

$$M_{L3} = 29 \times 3a - 8(a + 2a) - 3\left(\frac{a}{2} + \frac{3a}{2} + \frac{5a}{2}\right)$$

$$= 49{\cdot}5a \qquad ,, = 99/\sqrt{3}$$

The corresponding moments in the right hand half of the truss will be the same owing to uniform loading.

For shear stresses,

$29 \operatorname{cosec} \theta = 58/\sqrt{3}$
$26 \operatorname{cosec} \theta = 52/\sqrt{3}$
$18 \operatorname{cosec} \theta = 36/\sqrt{3}$
$15 \operatorname{cosec} \theta = 30/\sqrt{3}$
$7 \operatorname{cosec} \theta = 14/\sqrt{3}$
$4 \operatorname{cosec} \theta = 8/\sqrt{3}$

Table of Stresses

Compression + ; tension -

Member	Stress	Member	Stress
U_1U_2	$+55/\sqrt{3}$	L_0L_1	$-29/\sqrt{3}$
U_2U_3	$+88/\sqrt{3}$	L_1L_2	$-73/\sqrt{3}$
U_3U_4	$+99/\sqrt{3}$	L_2L_3	$95/\sqrt{3}$
U_4U_5	$+88/\sqrt{3}$	L_3L_4	$95/\sqrt{3}$
U_5U_6	$+55/\sqrt{3}$	L_4L_5	$73/\sqrt{3}$
		L_5L_6	$-29/\sqrt{3}$
U_1L_1	$52/\sqrt{3}$	L_0U_1	$+58/\sqrt{3}$
U_2L_2	$30/\sqrt{3}$	L_1U_2	$+36/\sqrt{3}$
U_3L_3	$-8/\sqrt{3}$	L_2U_3	$+14/\sqrt{3}$
L_3U_4	$-8/\sqrt{3}$	U_4L_4	$+14/\sqrt{3}$
L_4U_5	$-30/\sqrt{3}$	U_5L_5	$+36/\sqrt{3}$
L_5U_6	$-52/\sqrt{3}$	U_6L_6	$+58/\sqrt{3}$

5. Pratt truss: With inclined chords, loaded as shown.

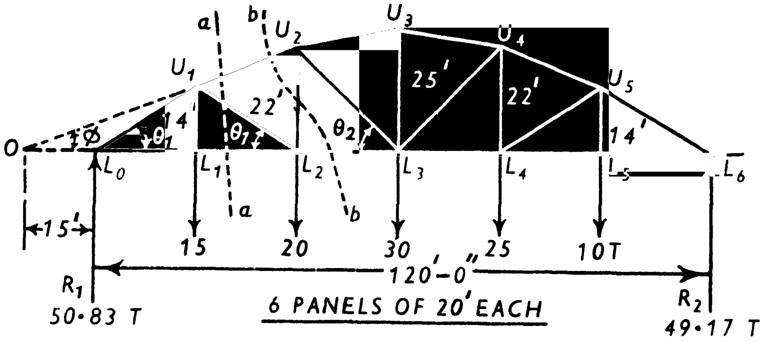


FIG. 199

The reactions at the supports are $R_1 = 50{\cdot}8$ tons and $R_2 = 49{\cdot}2$ tons.

$$\tan\,\theta_1 = \frac{14}{20} = \cdot 7$$

$$\therefore \quad \theta_1 = 35^\circ$$

$$\sin\,\theta_1 = \cdot 5736$$

$$\cos\,\theta_1 = \cdot 8192$$

$$\tan\,\theta_2 = \tfrac{22}{20} = 1\cdot 1$$

$$\therefore \quad \theta_2 = 47^\circ\ 44'$$

$$\sin\,\theta_2 = \cdot 74$$

$$\cos\,\theta_2 = \cdot 6726.$$

We shall work out the stresses in the members of the panel L_1L_2. If φ is the inclination of U_1U_2 with the horizontal,

$$\tan\,\varphi = \frac{22-14}{20} = \cdot 40$$

$$\therefore \quad \varphi = 21^\circ\ 48'$$

$$\sin\,\varphi = \cdot 3714$$

$$\cos\,\varphi = \cdot 9285.$$

Produce U_2U_1 to meet L_6L_0 at O. Then, from similar triangles,

$$\frac{OL_2}{U_2L_2} = \frac{20}{22-14} = \frac{20}{8}$$

$$\therefore \quad OL_2 = \frac{20}{8} \times 22 = 55 \text{ feet.}$$

$$\therefore \quad OL_0 = 55 - 40 = 15 \text{ feet.}$$

Section aa.

Taking moments about U_1,

$$l_1l_2 \times 14 = 50\cdot 8 \times 20$$

$$\therefore \quad l_1l_2 = \frac{508}{7} = 72\cdot 6 \text{ tons, tensile.}$$

Taking moments about L_2,

$$u_1u_2 \times OL_2 \sin\varphi = 50\cdot 8 \times 40 - 15 \times 20$$

$$\therefore \quad u_1u_2 \times 55 \times \cdot 3714 = 1732$$

$$\therefore \quad u_1u_2 = \frac{1732}{55 \times \cdot 3714} = 84\cdot 76 \text{ tons, compressive.}$$

Taking moments about O,

$$u_1l_2 \times OL_2 \sin\theta_1 = 50{\cdot}8 \times 15 - 15 \times 35 = 237$$

$$\therefore \quad u_1l_2 = \frac{237}{55 \times {\cdot}5736} = 7{\cdot}5 \text{ tons, tensile.}$$

Section bb.

For stress in the vertical U_2L_2, take moments about O. We have,

$$u_2l_2 \times OL_2 = 50{\cdot}8 \times 15 - 15 \times 35 - 20 \times 55 = -863$$

$$\therefore \quad u_2l_2 = \frac{863}{55} = 15{\cdot}7 \text{ tons, tensile.}$$

The student should complete the calculations for stresses in the remaining members, as an exercise.

6. Cantilever truss: Warren type, loaded as shown.

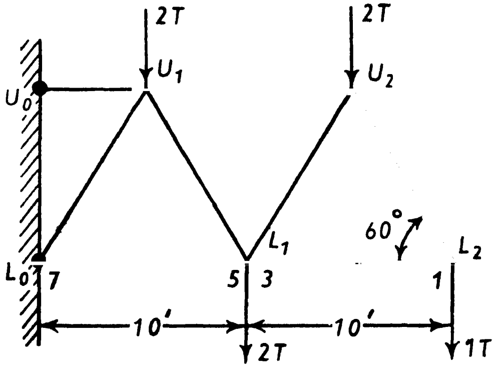


FIG. 200

$\theta = 60^\circ$

$\sin\theta = \sqrt{3}/2$

$\operatorname{cosec}\theta = 2/\sqrt{3}$

$M_{U2} = 1 \times \frac{a}{2}$ $\qquad \div h = 1/\sqrt{3}$

$M_{L1} = 1 \times a + 2 \times \frac{a}{2} = 2a$ $\qquad$,, $= 4/\sqrt{3}$

$M_{U1} = 1 \times \frac{3a}{2} + 2\left(a + \frac{a}{2}\right) = 4{\cdot}5a$ $\qquad$,, $= 9/\sqrt{3}$

$M_{L0} = 1 \times 2a + 2\left(\frac{a}{2} + a + \frac{3a}{2}\right) = 8a$ $\qquad$,, $= 16/\sqrt{3}$

The shears in the panels L_0L_1 and L_1L_2 have been noted in fig. 200.

$$1 \times \operatorname{cosec} \theta = 2/\sqrt{3}$$
$$3 \times \operatorname{cosec} \theta = 6/\sqrt{3}$$
$$5 \times \operatorname{cosec} \theta = 10/\sqrt{3}$$
$$7 \times \operatorname{cosec} \theta = 14/\sqrt{3}$$

The stresses may be tabulated as under:

Table of Stresses

Compression +; tension—

Member	Stress	Member	Stress
U_0U_1	$-16/\sqrt{3}$	L_0L_1	$+9/\sqrt{3}$
U_1U_2	$-4/\sqrt{3}$	L_1L_2	$+1/\sqrt{3}$
U_1L_1	$-10/\sqrt{3}$	L_1U_2	$+6/\sqrt{3}$
U_2L_2	$-2/\sqrt{3}$	L_0U_1	$+14/\sqrt{3}$

7. Cantilever truss: Triangular type, loaded as shown.

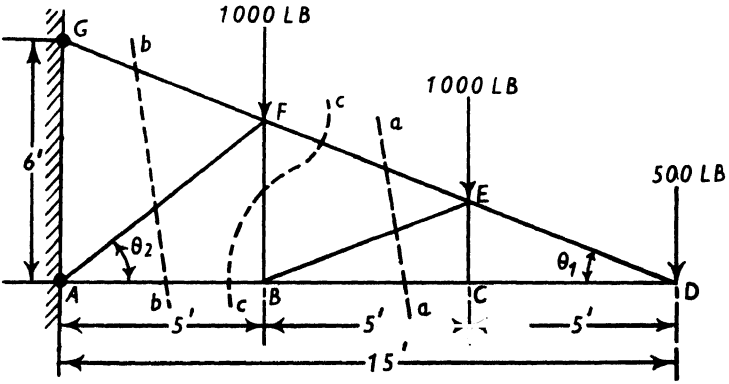


FIG. 201

$$\tan\theta_1 = \tfrac{2}{5} = \cdot 4$$

$$\therefore \quad \theta_1 = 21^\circ 48'$$

$$\sin\theta_1 = \cdot 3714$$

$$\cos\theta_1 = \cdot 9285$$

$$\tan\theta_2 = \tfrac{4}{5} = \cdot 8$$

$$\therefore \quad \theta_2 = 38^\circ\ 40'$$

$$\sin\theta_2 = \cdot 6248$$

$$\cos\theta_2 = \cdot 7608$$

Panel CD.

$$de \sin\theta_1 = 500$$

$$\therefore \quad de = 500 \operatorname{cosec}\theta_1 = \frac{500}{\cdot 3714} = 1346 \text{ lb., tensile.}$$

$$cd = de \cos\theta_1 = 500 \cot\theta_1$$

$$= 500 \times \tfrac{5}{2} = 1250 \text{ lb., compressive.}$$

Panel BC.

Section aa.

Taking moments about E,

$$bc \times 2 = 500 \times 5$$

$$\therefore \quad bc = 1250 \text{ lb., compressive.}$$

Taking moments about B,

$$ef \times BD \sin\theta_1 = 500 \times 10 + 1000 \times 5 = 10000$$

$$\therefore \quad ef \times 10 \times \cdot 3714 = 10000.$$

$$\therefore \quad ef = \frac{1000}{\cdot 3714} = 2692 \text{ lb., tensile.}$$

Taking moments about D,

$$be.BD.\ \sin\theta_1 = 1000 \times 5$$

$$\therefore \quad be \times 10 \times \cdot 3714 = 5000$$

$$\therefore \quad be = 1346 \text{ lb., compressive.}$$

$$ce = 0.$$

Panel AB.

Section bb.

Taking moments about F,

$$ab \times 4 = 500 \times 10 + 1000 \times 5$$

$$\therefore \quad ab = 2500 \text{ lb., compressive.}$$

Taking moments about A,

$$gf \times AD \sin\theta_1 = 500 \times 15 + 1000 \times 10 + 1000 \times 5$$
$$gf \times 15 \times \cdot 3714 = 22500$$
$$gf = \frac{22500}{15 \times \cdot 3714} = 4038 \text{ lb., tensile.}$$

Taking moments about D,

$$af \times AD \sin\theta_2 = 1000 \times 5 + 1000 \times 10$$
$$af \times 15 \times \cdot 6248 = 15000$$
$$af = \frac{15000}{15 \times \cdot 6248} = 1600 \text{ lb., compressive.}$$

Section cc.

Taking moments about D,

$$bf \times 10 = 1000 \times 5$$
$$\therefore \quad bf = 500 \text{ lb., compressive.}$$

The stresses in all the members of the truss have been determined.

8. Steel-trestle, with two tiers, loaded as shown:

If θ is the inclination of AB and A_1B_1 to the vertical,

$$\tan\theta = \frac{4}{24} = \cdot 1667$$
$$\therefore \quad \theta = 9^\circ\ 28'$$
$$\therefore \quad \sin\theta = \cdot 1645$$
$$\cos\theta = \cdot 9864$$
$$\sec\theta = 1\cdot 014.$$

(i) Vertical loading, as shown.

Since the vertical loads are symmetrically arranged, the stresses in the diagonals BA_1 and CB_1 will be zero.

$$ab = a_1b_1 = 40\sec\theta = 40 \times 1\cdot 014 = 40\cdot 56 \text{ tons, compressive}$$
$$aa_1 = ab \sin\theta = 40 \tan\theta = \tfrac{40}{6} = 6\cdot 67 \text{ tons, compressive.}$$
$$bc = b_1c_1 = (40 + 3) \sec\theta = 43 \times 1\cdot 014$$
$$= 43\cdot 60 \text{ tons, compressive.}$$
$$bb_1 = 3 \tan\theta = 3 \times \tfrac{1}{6} = \tfrac{1}{2} \text{ ton, compressive}$$

(ii) Horizontal loading, as shown.

Top-tier.

$aa_1 = 2$ tons, compressive.

$ab = 0$.

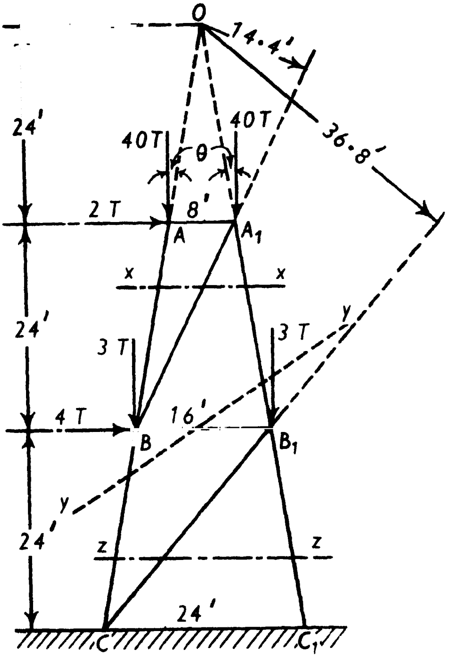


FIG. 202

Section xx.

Taking moments about B.

$a_1b_1 \times BB_1 \cos\theta = 2 \times 24$

$\therefore \quad a_1b_1 \times 16 \times \cdot9864 = 48$

$\therefore \quad a_1b_1 = 3 \times 1\cdot014 = 3\cdot042$ tons, compressive.

Taking moments about O,

$ba_1 \times 14\cdot4 = 2 \times 24$

$\therefore \quad ba_1 = \dfrac{48}{14\cdot4} = 3\cdot33$ tons, tensile.

Section yy.

Taking moments about O,

$bb_1 \times 48 = 2 \times 24 + 4 \times 48 = 240$

$\therefore \quad bb_1 = 5$ tons, compressive.

Bottom tier.

Section zz.

Taking moments about B_1,

$bc \times BB_1 \cos\theta = 2 \times 24$

$$\therefore \quad bc = \frac{48}{16 \times \cdot 9864} = 3 \times 1{\cdot}014 = 3{\cdot}042 \text{ tons, tensile.}$$

Taking moments about C,

$b_1c_1 \times 24 \cos\theta = 2 \times 48 + 4 \times 24 = 192$

$$\therefore \quad b_1c_1 = \frac{192}{24 \times \cdot 9864} = 8 \times 1{\cdot}014$$

$$= 8{\cdot}112 \text{ tons, compressive.}$$

Taking moments about O,

$cb_1 \times 36{\cdot}8 = 2 \times 24 + 4 \times 48 = 240$

$$\therefore \quad cb_1 = \frac{240}{36{\cdot}8} = 6{\cdot}52 \text{ tons, tensile.}$$

The stresses in all the members have thus been evaluated. It will be noticed that a trestle is just a vertical cantilever in so far as horizontal loading is concerned.

9. K-truss, loaded as shown: This truss is a perfect frame and there is no difficulty about calculating the stresses in the members.

We shall investigate the stresses in the members of the panel L_2L_3. Let us first study the joint M_2. There are four forces acting at the joint, two along the vertical U_2L_2 and two along the diagonals M_2U_3 and M_2L_3. Resolving horizontally,

horizontal component of m_2u_3 = horiz. component m_2l_3. If M_2 is placed at the middle of the vertical, the inclination θ of either diagonal to the horizontal is the same being

$\tan^{-1}\frac{15}{20}$ in this case. It is evident that the stresses m_2u_3 and m_2l_3 will, therefore, be equal, m_2u_3 being a push and m_2l_3 an equal pull.

Now, study the *section aa.*

Taking moments about U_2

$l_2l_3 \times 30 = 45 \times 40 - 10 \times 20 = 1600$

$\therefore \quad l_2l_3 = 53{\cdot}33$ tons, tensile.

$u_2u_3 = 53{\cdot}33$ tons, compressive, by taking moments about L_2.

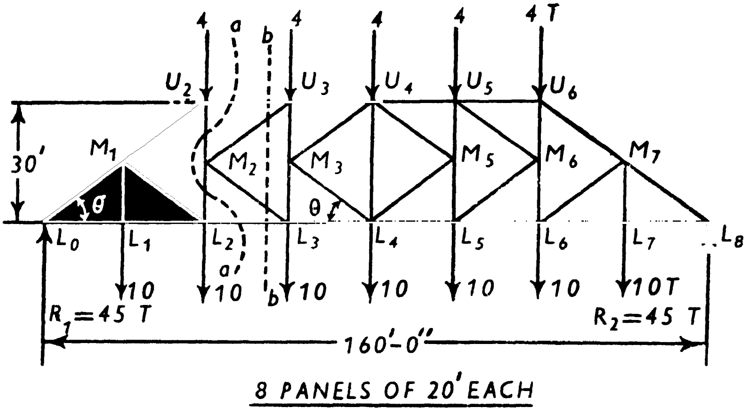


FIG. 203

Section bb.

Resolving vertically,

$m_2l_3 \sin\theta + m_2u_3 \sin\theta = 45 - 10 - 10 - 4 = 21$ tons, being the shear F in the panel.

Since $m_2l_3 = m_2u_3$ numerically, we have

$m_2l_3 = m_2u_3 = \frac{1}{2}F\operatorname{cosec}\theta = \frac{21}{2} \times \frac{5}{3} = 17{\cdot}5$ tons.

At the joint L_3, resolving the forces vertically,

$m_3l_3 + 10 = m_2l_3 \sin\theta = \frac{1}{2} \times 21$

$\therefore \quad m_3l_3 = 0{\cdot}5$ tons, compressive.

Finally, at the joint U_3, resolving the forces vertically,

$u_3m_3 + 4 = m_2u_3 \sin\theta = \frac{1}{2} \times 21$

$\therefore \quad u_3m_3 = 6{\cdot}5$ tons, tensile.

The computation of stresses in the remaining members is left as an exercise.

10. Space frames: Problems on elementary space frames like sheer legs, tripods, etc., are easy to solve. The method essentially consists in reducing three-dimensional forces at a joint to just *three* forces by replacing some of the forces by a resultant, if necessary.

If a point is in equilibrium under the action of three forces, they must necessarily be coplanar. Let P_1, P_2 and P_3 be three forces actions at a point O, along OA, OB and OC respectively and keeping it in equilibrium. We can reduce the system to two forces, viz., (i) P_1 acting along OA and (ii) a force R, which is the resultant of P_2 and P_3. R must pass through O and its line of action must lie in the plane of OB and OC. Since P_1 and R keep the point O in equilibrium, they must be equal and opposite and act along the same straight line, i.e. OA must lie in the plane of OB and OC. The proposition is thus proved. The statical problem of three forces at a point is then solved by the use of the triangle of forces or by using Lamé's formula, $\frac{P_1}{\sin\alpha_1} = \frac{P_2}{\sin\alpha_2} = \frac{P_3}{\sin\alpha_3}$, where α_1, α_2 and α_3 are the angles between the lines of action of the forces $P_2 - P_3$, $P_3 - P_1$ and $P_1 - P_2$ respectively.

11. Sheer legs: Each leg of a pair of sheer legs is 15 feet long. The feet are placed 12 feet apart, the line joining them being 24 feet from the foot of the guy-rope. If the guy-rope is 32 feet long, find the thrust in each leg and the pull in the guy-rope when a load of 10 tons is suspended from the head. What is the overhang of the load?

Let OA and OB represent the sheer legs and OC the guy-rope. See fig. 204(*i*). The head O is in equilibrium under four forces, (i) the load W, acting vertically, (ii) the thrust P_1 along the leg AO, (iii) the thrust P_2, along the leg BO and (iv) the pull P_3 acting along the guy-rope OC.

By symmetry $P_1 = P_2$ and these thrusts can be replaced by their resultant R acting along DO, in the plane BOA, D being the middle point of AB.

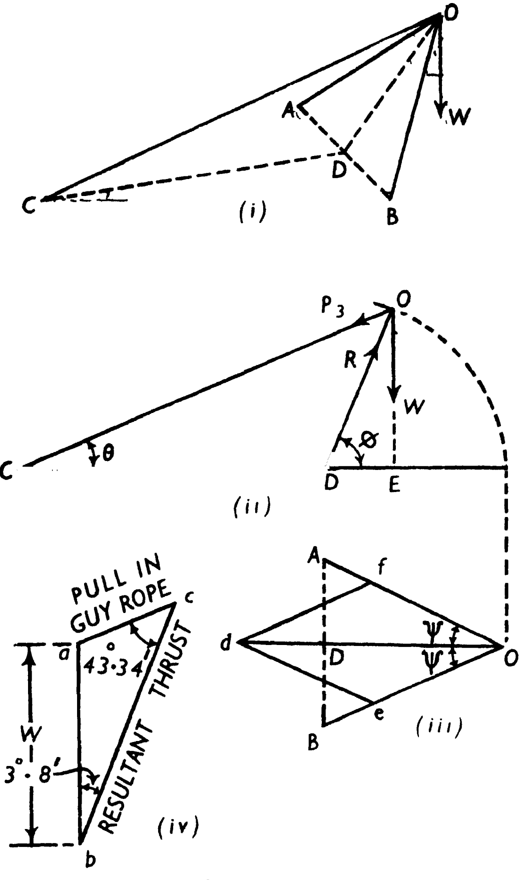


Fig. 204

The system of forces now reduces to three, viz., the vertical load W, the pull P_3 of the guy-rope and the resultant R. These must necessarily lie in one plane, which is the vertical plane through OC, as shown in fig. 204(*ii*).

To construct the triangle COD, the two sides CD and OC have been given. $CD = 24$ feet and OC, the length of the guy-rope is 32 feet. The remaining side $OD = \sqrt{OA^2 - AD^2} =$

$\sqrt{15^2 - 6^2} = \sqrt{189} = 13{\cdot}75$ feet. The triangle *COD* can now be constructed.

The length *OD* can also be obtained graphically as shown in fig. 204(*iii*). Set off $AB = 12$ feet to scale, with $OA = OB = 15$ feet to the same scale. Join *O* to *D*, the middle point of *AB*. Use the length *OD* so obtained to construct the triangle *COD* in fig. 204(*ii*), in which three forces are acting at *O*, viz., the vertical load $W = 10$ tons, the pull P_3 in the guy-rope *OC* and the resultant thrust *R* along *DO*.

Draw the triangle of forces *abc*, as in fig. 204(*iv*). *ab* represents *W*, *bc* the thrust *R* and *ca* the pull P_3. Scale off P_3 which is 5·7 tons, while *R* is 13·3 tons.

To get the components P_1 and P_2 of *R* along *AO* and *BO*, set off *Od* [along *OD* in fig. 204(*iii*)] to scale, to represent 13·3 tons. Complete the parallelogram *Oedf*, with *Od* as diagonal. Then $P_1 = Of$ to scale and $P_2 = Oe$ to scale. Each thrust is 7·27 tons.

The overhang of the load is given by *DE* in fig. 204(*ii*). It is 5·4 feet.

The results could have been obtained by calculation as under:

In fig. 204(*ii*),

$$\cos\theta = \frac{OC^2 + CD^2 - OD^2}{2.OC.OCD} = \frac{32^2 + 24^2 - 189}{2.32.24} = \frac{1411}{1536}$$

$$= {\cdot}9184$$

$$\therefore \quad \theta = 23^\circ\ 18' \text{ and } 90 \quad \theta = 66^\circ\ 42'$$

Also,

$$CD + DE = CE$$

$$\therefore \quad CD + OD\cos\varphi = OC.\cos\theta$$

$$\text{i.e.} \quad 24 + 13{\cdot}75\cos\varphi = 32\cos\theta$$

$$\therefore \quad 13{\cdot}75\cos\varphi = 32 \times {\cdot}9184 - 24 = 29{\cdot}4 - 24 = 5{\cdot}4$$

$$\therefore \quad \cos\varphi = \frac{5{\cdot}4}{13{\cdot}75} = {\cdot}3927$$

$$\therefore \quad \varphi = 66^\circ\ 52' \text{ and } 90 - \varphi = 23^\circ\ 8'$$

$$\therefore \quad \varphi - \theta = 43^\circ\ 34'.$$

Using Lamé's theorem,

$$\frac{P_3}{\sin 23^\circ\ 8'} = \frac{R}{\sin 66^\circ\ 42'} = \frac{W}{\sin 43^\circ\ 34'}$$

$$\therefore \qquad P_3 = \frac{10 \times \cdot 3927}{\cdot 6892} = 5\cdot 70 \text{ tons}$$

$$\therefore \qquad R = \frac{10 \times \cdot 9184}{\cdot 6892} = 13\cdot 32 \text{ tons.}$$

The components of R along AO and BO will be given by,

$$R = 2P_1 \cos\Psi, \text{ where } \angle AOD = \angle BOD = \Psi.$$

$$\sin\Psi = \frac{AD}{AO} = \frac{6}{15} = \cdot 4$$

$$\therefore \qquad \Psi = 23^\circ\ 36'$$

and $\cos\Psi = \cdot 9164$

$$\therefore \quad 13\cdot 32 = 2P_1 \times \cdot 9164$$

$$\therefore \qquad P_1 = 7\cdot 27 \text{ tons} = P_2, \text{ by symmetry.}$$

The overhang $DE = OD \cos\varphi = 5\cdot 4$ feet.

12. Derrick crane: The vertical post *OA* of a derrick crane is 15 feet high and is supported by ties *OB* and *OC* so that the feet *A*, *B* and *C* form an equilateral triangle, on level ground, each side being 20 feet. The jib *AD* is 24 feet long, while the tie *OD* is 14 feet. A load of 10 tons is suspended from *D*. Calculate the stresses in the members when the plane *OAD* is inclined at 15° to its central position.

Let *OF* be the trace of the plane *OAD* on the plane *BOC* and let *AF* be its trace on the plane *ABC*. If *E* is the middle point of *BC*, *AF* makes an angle of 15° with *AE*. See fig. 205(*iii*).

Let P_1, P_2, P_3, P_4 and P_5 be the stresses in the vertical post *OA*, the ties *OB* and *OC*, the tie *OD* and the jib *AD*, respectively, due to the load $W = 10$ tons at the crane-head *D*.

The joint *O* is in equilibrium under the pull P_4 in *OD*, the thrust P_1 in *AO* and the pulls P_2 and P_3 along the ties *BO* and CO respectively. P_2 and P_3 may be replaced by a single resultant R, which must necessarily lie in the plane of its components. viz., plane BOC.

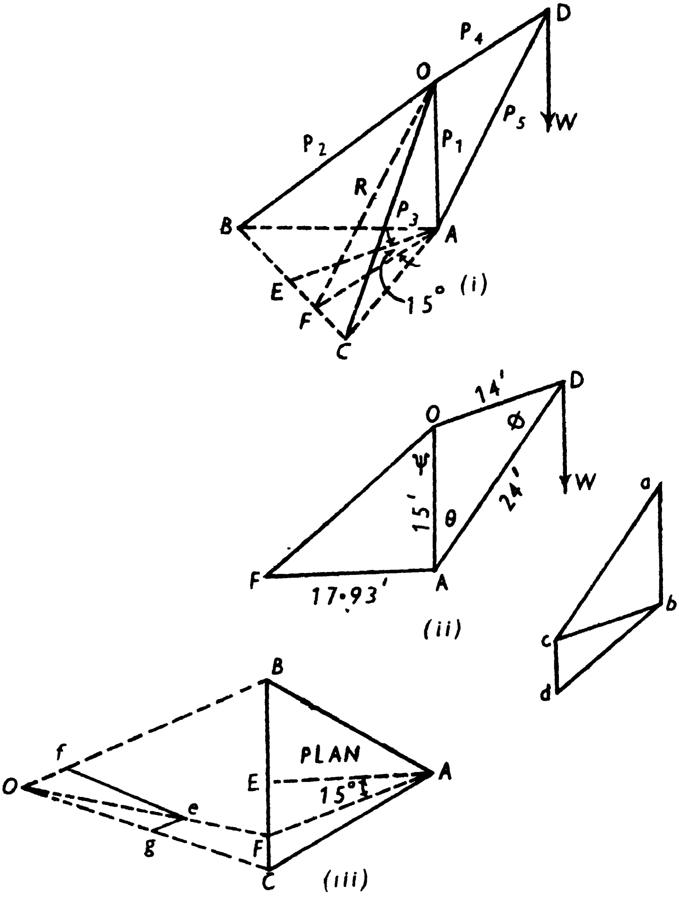


Fig. 205

The joint O is now in equilibrium under the three forces, P_4, P_1 and R and these must lie in the same plane, viz., the plane AOD. The line of action of R, the resultant of P_2 and P_3, is therefore the trace OF of the planes BOC and AOD.

In fig. 205(*ii*), the lines of action of the forces have been set off in the plane *FODA*. *OA* is 15 ft.; *OD* = 14 ft. and *AD* = 24 feet. *AF* may be scaled off from fig. 205(*iii*) or calculated. $AE = 10\sqrt{3} = 17{\cdot}32$ feet and $AF = AE \sec 15^\circ = \dfrac{17{\cdot}32}{{\cdot}9659}$ $= 17{\cdot}93$ feet.

Construct the stress-diagram *abdc* for the joints *D* and *O*.

For the joint *D*, $ab = W = 10$ tons; *bc* represents the pull P_4 in the tie *OD* and *ca* represents the push P_5 in the jib. From the stress diagram, $P_4 = 9{\cdot}33$ tons and $P_5 = 16$ tons.

For the joint *O*, *cb* represents the pull P_4 in the tie *OD*. *bd* represents the resultant pull *R* and *dc* represents the thrust P_1 in the vertical post *AO*. From the diagram, $R = 11{\cdot}3$ tons and $P_1 = 3{\cdot}8$ tons.

To split up *R* into its two components P_2 and P_3 along *BO* and *CO* respectively, construct the triangle *BOC*, with *BC* = 20 feet as base and *BO* = *OC* = 25 feet, which is the length of each tie. See fig. 205(*iii*). Join *OF*. Along *OF*, set off *Oe* = 11·3 tons to scale. Complete the parallelogram *Ofeg* with *Oe* as the diagonal. *Of* represents P_2 and *Og* represents P_3.

The stresses in all the members have thus been found.

They can be calculated as under:

In fig. 205(*ii*), let $\angle OAD = \theta$, $\angle ODA = \varphi$ and $\angle FOA = \Psi$.

Then,

$$\cos\theta = \frac{15^2 + 24^2 - 14^2}{2.15.24} = \frac{605}{720} = {\cdot}8403$$

$$\therefore \quad \theta = 32^\circ\ 50'$$

$$\cos\varphi = \frac{14^2 + 24^2 - 15^2}{2.14.24} = \frac{547}{672} = {\cdot}8139$$

$$\therefore \quad \varphi = 35^\circ\ 31'$$

$$\therefore \quad \theta + \varphi = 68^\circ\ 21'.$$

Also, $\tan\Psi = \dfrac{AF}{OA} = \dfrac{17{\cdot}93}{15} = 1{\cdot}195$

$\therefore \quad \Psi = 50^\circ\ 5'$

At the joint D,

$$\frac{W}{\sin\varphi} = \frac{P_4}{\sin\theta} = \frac{P_5}{\sin(\theta+\varphi)}$$

$$\therefore \quad P_4 = \frac{10 \times {\cdot}5422}{{\cdot}5809} = 9{\cdot}335 \text{ tons}$$

$$\text{and} \quad P_5 = \frac{10 \times {\cdot}9294}{{\cdot}5809} = 16 \text{ tons.}$$

At the joint O,

$$\frac{P_1}{\sin 161^\circ\ 44'} = \frac{P_4}{\sin 50^\circ\ 5'} = \frac{R}{\sin 111^\circ\ 39'}$$

$$\therefore \quad P_1 = \frac{9{\cdot}335 \times {\cdot}3133}{{\cdot}7670} = 3{\cdot}81 \text{ tons.}$$

$$R = \frac{9{\cdot}335 \times {\cdot}9294}{{\cdot}7670} = 11{\cdot}31 \text{ tons.}$$

13. Tripod: *OA*, *OB*, *OC*, of length 10, 11 and 12 feet respectively form the legs of a tripod, with the feet resting on level ground so that $AB = 8$ feet; $BC = 9$ feet and $CA = 10$ feet. Find the thrust in each leg when a load of 10 tons is suspended from O.

The point O is in equilibrium under the vertical load W and the thrusts P_1, P_2 and P_3 along the legs AO, BO and CO respectively. These four forces can be reduced to three viz., (i) the thrust P_1 along AO, (ii) the vertical load W and (iii) the resultant R of the thrusts P_2 and P_3.

The line of action of R must lie in the plane BOC, R being the resultant of P_2 and P_3. Being one of the three force keeping the point O in equilibrium, its line of action must also lie in the plane of the other two forces, viz., P_1 and W, i.e. in the vertical plane through AO.

Its line of action is therefore OE, the trace of the vertical plane through AO on the plane BOC.

To obtain *OE*, it is sufficient to get the projection *D* of the point *O* on the ground. We can then join *AD* and produce it to meet *BC* at *E*, thus fixing *E*.

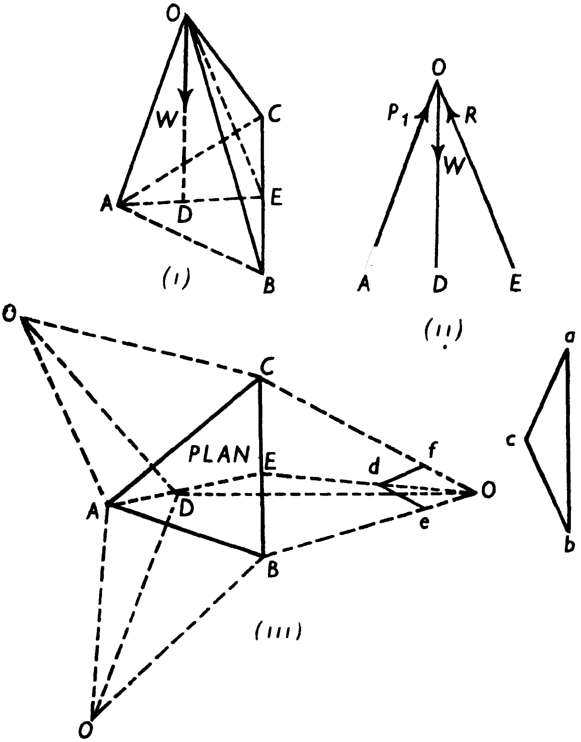


FIG. 206

Construct the triangle *ABC*, making *AB* = 8 feet, *BC* = 9 feet and *CA* = 10 feet, to some convenient scale, as in fig. 206(*iii*). The triangle *ABC* is the plan of the feet of the three legs of the tripod on level ground. Now swing the plane *BOC* about *BC* until it lies level on the ground. In other words, construct the triangle *BOC*, making *BO* = 11 feet and *CO* = 12 feet, to the same scale.

Likewise, swing over the planes *COA* and *AOB* on to level ground about the axes *CA* and *AB* respectively. The three planes are shown dotted in fig. 206(*iii*).

If we, now, pick up any one of these planes, say *BOC* and swing it back to its original position about *BC*, the path traced out by the projection of *O* on the ground will be the perpendicular from *O* on *BC*. The locus of *D* is, thus, the perpendicular from *O* on *BC*. Similarly *D* lies on the perpendicular from *O* (of the triangle *OAB*) on *AB*, or from *O* (of the triangle *OAC*) on *AC*. The point *D* is thus located and so is *E*.

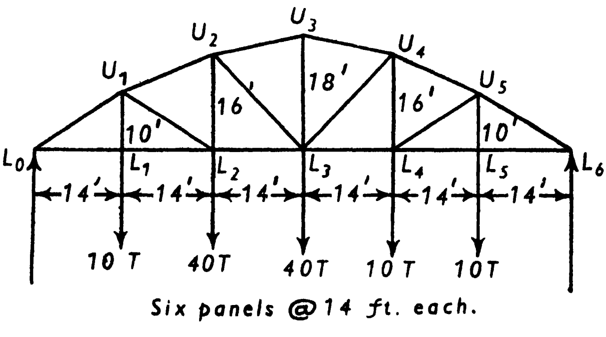
Next, construct the plane *AOE*. On *ADE* as base, construct the triangle *AOE* as in fig. 206(*ii*). Through *D*, draw the perpendicular to *AE*. With centre *A* and radius equal to $AO = 10$ feet in this case, strike off an arc cutting the perpendicular through *D* at *O*. Join *OE*.

W acts along *OD*; P_1 acts along *AO*; and *R*, the resultant of P_2 and P_3 acts along *EO*. Draw *abc* the triangle of forces, *ab* representing the load $W = 10$ tons. Then *bc* represents *R* and *ca* represents the thrust P_1.

Along *OE* in fig. 206(*iii*), set off *Od* to scale to represent *R*. To get its components along *BO* and *CO*, complete the parallelogram *Oedf*. Then *eO* represents P_2 and *fO* represents P_3.

EXAMPLES XIV

(1) The dimensions and loading of a bridge truss are shown in fig. 207. Using an analytical method, find the forces in the four members meeting at the joint U_2. (L. U.)



Six panels @ 14 ft. each.

FIG. 207

$\left\{ \begin{array}{l} u_2u_1 = 104{\cdot}7 \text{ tons, compression; } u_2l_2 = 31{\cdot}25 \text{ tons, tension;} \\ u_2l_3 = 4{\cdot}43 \text{ tons, compression; } u_2u_3 = 94{\cdot}23 \text{ tons, compression.} \end{array} \right\}$

(2) Find the loads in members *CF* and *FG* of the pin-jointe frame shown in fig. 208, when a unit vertical load acts at each c the joints *B*, *C* and *D* and half a unit vertical load at *A* and *E*.

The frame is symmetrical about the vertical centre line; *A*, *C*, *D*, *E* are equally spaced and *BH*, *CG*, *DF* are perpendicular *AE*. (L. U

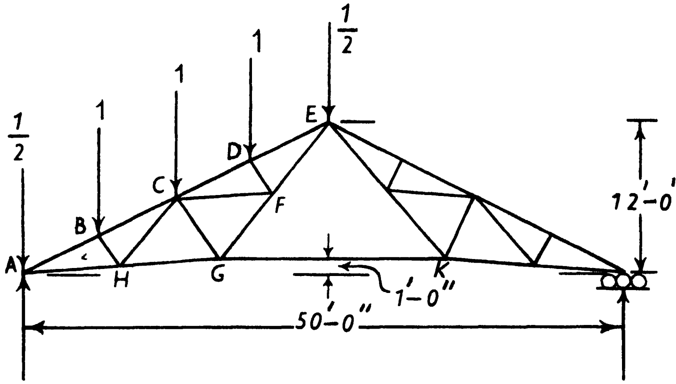


FIG. 208

{ $gk = 2{\cdot}273$ tons, tension; $cf = 1{\cdot}214$ tons, tension.
$fg = 2{\cdot}65$ tons, tension. }

(3) Fig. 209 shows a pin-jointed cantilever crane, the tensioı in the cable being one ton. Using the method of sections, calculat the forces in the members *AB*, *AG* and *GH* and state whether the are tensile or compressive. (L. U

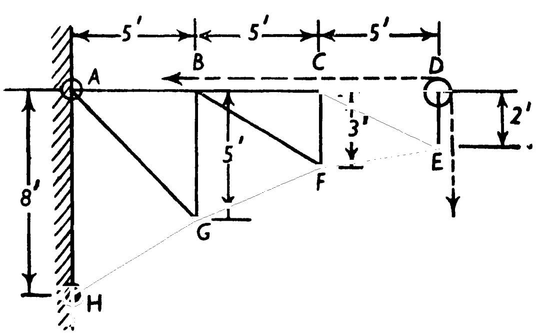


FIG. 209

{ $ab = 1$ ton, tensile; $ag = {\cdot}177$ ton, compressive;
$gh = 2{\cdot}186$ tons, compressive. }

(4) The braced frame shown in fig. 210 is simply supported at *J* and *F* and carries four 1-ton loads at *B*, *C*, *D* and *E*.

Determine the forces in the members *CJ*, *CD*, *CH* and *JH*.

(L. U.)

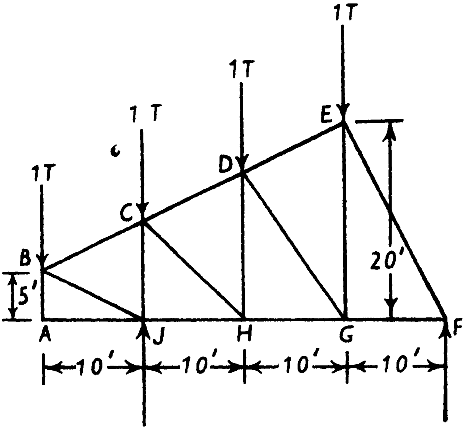


FIG. 210

{ *cj* = 2·83 tons, compressive; *cd* = ·2484 ton, compressive;
ch = 1·728 tons, tensile; *jh* = 1 ton, compressive. }

(5) Determine by the method of sections or any other analytical method, the forces in the five members meeting at the joint *G* of the truss shown in fig. 211. (L. U.)

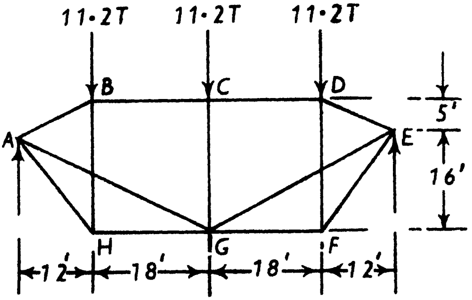


FIG. 211

{ *gh* = *gf* = 3·9 tons, tensile.
ga = *ge* = 11·9 tons, tensile.
gc = 11·2 tons, compressive. }

(6) The roof truss shown in fig. 212 is symmetrical about centre-line; the main tie is horizontal and the height of the trus 12′ 6″. The rafters are divided into four equal lengths by the stru which are perpendicular to the rafters. The joint at *A* is to regarded as hinged to the support while that at *B* is to be consider as resting on a roller. The vertical loads are downward and t inclined loads are perpendicular to the rafters and upward; all loa are in pounds. Determine by calculation: (a) the reactions a *A* and *B* in magnitude and direction (b) the forces in the members *CE* and *EF* stating for each whether it is tensile or compressive.

(L. U

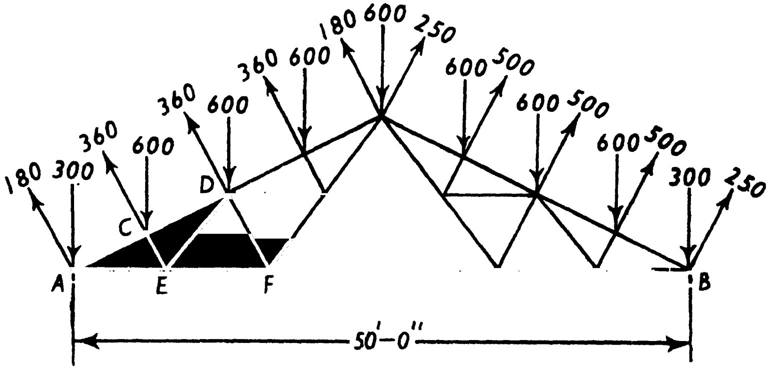


FIG. 212

$\{V_A = 956$ lb. ↑; $H_A = 250$ lb., ←
$V_B = 767$ lb. ↑; $ce = 176{\cdot}64$ lb. compressive.
$ef = 1768$ lb., tensile. $\}$

(7) A load of 200 lb. hangs supported by three wires *AD*, *BD* and *CD*, the point of attachment being *D*. The ends *A*, *B* and *C* of the wire form an equilateral triangle of side 6 feet in a horizontal plane. The length of *AD* is 6 feet, of *BD* 5 feet and of *CD* 5′ 6″. Find the tension in each wire. (L. U.)

$\{ da = 56$ lb.; $db = 112$ lb.; $dc = 84$ lb. $\}$

(8) A tripod with three 5-feet legs having a common ball-join is set up on a uniform slope. The feet form an equilateral triangl with 6-ft. sides, with two of the feet in a line of steepest slope and having a difference in level of one foot. Find the load induced in each leg by a load of 100 lb. hung from the ball-joint. (L. U.)

{ 61, 34, 44 lb. }

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